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Spectral Pairs, Mixed Hodge Modules, and Series of Plane Curve Singularities

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Abstract. We consider a mixed Hodge module $\mathcal{M}$ on a normal surface singularity (X, x) and a holomorphic function germ $f : (X, x) \to (\mathbf{C}, 0)$. For the case that $\mathcal{M}$ has an abelian local monodromy group, we give a formula for the spectral pairs of f with values in $\mathcal{M}$. This result is applied to generalize the Sebastiani-Thom formula and to describe the behaviour of spectral pairs in series of singularities.

Contents

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1. Introduction

Spectral pairs were introduced first in [17] as discrete invariants of the mixed Hodge structure on the vanishing cohomology of an isolated hypersurface singularity. The spectral pairs which are considered in this article are defined following a slightly different convention, as in [11]. This invariant encodes the dimensions of the eigenspaces of the semisimple part T_s of the monodromy acting on each subquotient $Gr^W_{p+q}Gr^p_F$ of the vanishing cohomology, and takes its values in the group ring $\mathbf{Z}[\mathbf{Q} \times \mathbf{Z}]$.

Instead of vanishing cycles with constant coefficients one may consider vanishing cycles with coefficients in a mixed Hodge module [10]. We are led to consider these in the study of composed functions $f = p \circ \phi$ where $\phi : (X, x) \to (\mathbf{C}^2, 0)$ is a 2-parameter smoothing of an isolated complete intersection singularity and $p : (\mathbf{C}^2, 0) \to (\mathbf{C}, 0)$ is a holomorphic function germ. The main result of this article gives a formula for the spectral pairs for such p at 0 with values in a mixed Hodge module on $(\mathbf{C}^2, 0)$ in terms of a decorated graph associated with $p^{-1}(0) \cup \Delta$, where Δ is the discriminant of the mixed Hodge module, under the assumption that the latter has an abelian local monodromy group G. In fact, in the Main Theorem (5.1), $(\mathbf{C}^2, 0)$ has been replaced by an arbitrary normal surface singularity. The mixed Hodge module we consider gives rise to a limit mixed Hodge structure on which G acts [9] and this action is used as input for the formula. The assumption about abelian monodromy is always fulfilled in case the complement of Δ has abelian local fundamental group, e.g., when Δ has normal crossings. We obtain generalizations of the Sebastiani-Thom formula (the case where $\phi = f \times g$ with f and g isolated hypersurface singularities) in Section 11. We also obtain formulas describing the behaviour of the spectral pairs in certain series of singularities, which generalize [11], where the case of Yomdin's series was treated.

A quick review of mixed Hodge modules and vanishing cycle functors is given in Section 2, which also contains the definition of spectral pairs and their basic properties. The flavour of our result is described in Section 3 by reformulating the case of a 1-dimensional base. The ingredients of the main formula are defined in Section 4, whereas its statement and proof form the content of Section 5. Some illustrative examples are treated in Section 6. Sections 7–10 deal with the application to series of singularities.

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2. Mixed Hodge Modules and Spectral Pairs

Let X be a (separated and reduced) complex analytic space. In [10] the category $MHM(X)$ of mixed Hodge modules on X is associated with X. This category is stable under certain cohomological functors, for example under $\mathcal{H}^j f^*$ and $\mathcal{H}^j f^!$ associated with a morphism f of complex analytic spaces, and under $\mathcal{H}^j f_*$ associated with a projective (or proper Kähler) morphism f. Moreover, if g is a holomorphic function on X and $X_0 = g^{-1}(0)$, then the vanishing and the nearby cycle functors $\varphi_g, \psi_g : MHM(X) \to MHM(X_0)$ are defined. All these functors are compatible with the corresponding perverse cohomological functors on the underlying perverse sheaves via the forgetful (exact) functor

$$rat : MHM(X) \to Perv(\mathbf{Q}_X)$$

which assigns to a mixed Hodge module the underlying perverse sheaf (with $\mathbf{Q}$ coefficients). (For the definition of the functors φ_g and ψ_g at the level of the constructible sheaves, see [2].)

The vanishing and nearby cycle functors have a functor automorphism T_s of finite order. It is provided by the Jordan decomposition $T = T_s \cdot T_u$ of the monodromy T.

One has the decompositions:

$$\psi_g = \psi_{g,1} \oplus \psi_{g,\neq 1} \quad \text{respectively} \quad \varphi_g = \varphi_{g,1} \oplus \varphi_{g,\neq 1}$$

such that T_s is the identity on $\psi_{g,1}$ and $\varphi_{g,1}$ and has no 1-eigenspace on $\psi_{g,\neq 1}$ and $\varphi_{g,\neq 1}$. One has the canonical morphisms:

$$can : \psi_g \to \varphi_g \quad \text{and} \quad Var : \varphi_g \to \psi_g(-1),$$

compatible with the action of T_s, such that

$$can : \psi_{g,\neq 1} \xrightarrow{\sim} \varphi_{g,\neq 1} \tag{1}$$

is an isomorphism.

Let $D^b MHM(X)$ be the derived category of $MHM(X)$ (i.e., the category of bounded complexes whose cohomologies are mixed Hodge modules on X). Let $i : Y \to X$ be a closed immersion and $j : U \to X$ the inclusion of the complement of Y. Then the cohomological functors are lifted to functors $i_*, i^!, i^*, j^*, j_*, j_!$; and we have the functorial distinguished triangles for $\mathcal{M} \in D^b MHM(X)$:

$$\begin{aligned} &\to j_! j^* \mathcal{M} \to \mathcal{M} \to i_* i^* \mathcal{M} \xrightarrow{+1} \\ &\to i_* i^! \mathcal{M} \to \mathcal{M} \to j_* j^* \mathcal{M} \xrightarrow{+1} . \end{aligned} \tag{2}$$

The connection between the two sets of functors is the following. Set $X_0 = g^{-1}(0)$ and let $i : X_0 \to X$ be the corresponding immersion. Then for $\mathcal{M} \in Ob\, MHM(X)$ one has:

$$\begin{aligned} 0 \to \mathcal{H}^{-1} i^* \mathcal{M} \to \psi_{g,1} \mathcal{M} \xrightarrow{can} \varphi_{g,1} \mathcal{M} \to \mathcal{H}^0 i^* \mathcal{M} \to 0; \\ 0 \to \mathcal{H}^0 i^! \mathcal{M} \to \varphi_{g,1} \mathcal{M} \xrightarrow{Var} \psi_{g,1} \mathcal{M}(-1) \to \mathcal{H}^1 i^! \mathcal{M} \to 0; \end{aligned} \tag{3}$$

and $\mathcal{H}^{k-1} i^* \mathcal{M} = \mathcal{H}^k i^! \mathcal{M} = 0$ if $k \notin \{0, 1\}$.

On the other hand, if $f: X \to Y$ is a proper morphism and g is a holomorphic function on Y, then for any $\mathcal{M} \in Ob\, MHM(X)$ one has:

$$\psi_g \mathcal{H}^j f_* \mathcal{M} = \mathcal{H}^j f_* \psi_{g\circ f} \mathcal{M} \quad \text{(and similarly for } \varphi). \tag{4}$$

Example 2.1. Assume that X is smooth. A mixed Hodge module $\mathcal{M} \in MHM(X)$ is called smooth if $rat\mathcal{M}$ is a local system ([10]).

Example 2.2. The module $\mathcal{M}$ is called pure of weight n (or a polarizable Hodge module of weight n) if $Gr_i^W \mathcal{M} = 0$ for $i \neq n$.

The category of smooth polarizable mixed Hodge modules is equivalent to the category of variation of polarizable mixed Hodge structures which are admissible in the sense of [5].

Example 2.3. $MHM(\text{point})$ is the category of polarizable $\mathbf{Q}$-mixed Hodge structures ([10] (3.9)).

If g_1 and g_2 are two holomorphic functions such that $g_1^{-1}(0)$ intersects $g_2^{-1}(0)$ transversally along X_0, then

$$\psi_{g_1}\psi_{g_2} = \psi_{g_2}\psi_{g_1} : MHM(X) \to MHM(X_0) \quad \text{(the same for } \varphi\text{'s)}.$$

In this case $\psi_{g_1}\psi_{g_2}\mathcal{M}$ has two commuting monodromies T_1 and T_2 induced by the ψ-functors.

Moreover, consider the holomorphic functions $g_1, \dots, g_s$ such that $s = \dim X$ and the intersection $\cap_{i=1}^s g_i^{-1}(0)$ is a regular point $x \in X$, and the divisor $\cup_{i=1}^s g_i^{-1}(0)$ in a neighbourhood of x has normal crossings. Then on $\psi_{g_1} \cdots \psi_{g_s} \mathcal{M} \in MHM(\{x\})$ the commuting set of monodromies $T_1, \dots, T_s$ acts. We make the set of this type of objects more explicit. For the definition of mixed Hodge structures, see [1].

Definition 2.4. For any abelian group G we let $MHS(G)$ denote the category of representations

$$\rho : G \to Aut_{MHS}(H)$$

for H a mixed Hodge structure. For such ρ we let ρ^{pq} denote the induced representation of G in $Aut_{\mathbf{C}}(H^{pq})$, where $H^{pq} = Gr^W_{p+q} Gr^p_F H_{\mathbf{C}}$.

Example 2.5. Let $\mathcal{M}$ be a mixed Hodge module on X, $g : X \to \mathbf{C}$ holomorphic and $x \in g^{-1}(0)$. Then for all $j \in \mathbf{Z}$, we have the objects $\mathcal{H}^j i_x^* \psi_g \mathcal{M}$ and $\mathcal{H}^j i_x^* \varphi_g \mathcal{M}$ of $MHS(\mathbf{Z})$, where the action of $1 \in \mathbf{Z}$ is the semisimple part of the monodromy. By the monodromy theorem, this is an automorphism of finite order.

Definition 2.6. For $\rho : \mathbf{Z} \to Aut(H)$ in $MHS(\mathbf{Z})$ with finite order one defines:

$h^{pq}_\lambda :=$ multiplicity of $t - \lambda$ as a factor of the characteristic polynomial of $\rho^{pq}(1)$ (for $\lambda \in \mathbf{C}$);

and

$$Spp(\rho) = \sum_{\alpha, w} h^{[\alpha], w-[\alpha]}_{e^{2\pi i\alpha}} (\alpha, w) \in \mathbf{Z}[\mathbf{Q} \times \mathbf{Z}],$$

where $[\alpha]$ is the integral part of α. Moreover, for $g : X \to \mathbf{C}$ holomorphic and $x \in g^{-1}(0)$ one defines for a mixed Hodge module $\mathcal{M}$ on X:

$$Spp_\psi(\mathcal{M}, g, x) := \sum_j (-1)^j Spp\, \mathcal{H}^j i_x^* \psi_g \mathcal{M};$$

$$Spp_\varphi(\mathcal{M}, g, x) := \sum_j (-1)^j Spp\, \mathcal{H}^j i_x^* \varphi_g \mathcal{M}.$$

These take their values in $\mathbf{Z}[\mathbf{Q}\times\mathbf{Z}]$.

REMARK 2.7. In [17] the invariant $Spp_{St}(g,0)$ of spectral pairs was defined for an isolated hypersurface singularity $g:(\mathbf{C}^{n+1},0)\to(\mathbf{C},0)$. Its relation with the invariants above is as follows:

$$\text{if } \ Spp_{St}(g,0)=\sum n_{\alpha,w}(\alpha,w), \text{ then}$$
$$Spp_{\varphi}(\mathbf{Q}^H_{\mathbf{C}^{n+1}}[n+1],g,0)=\sum_{\alpha\notin\mathbf{Z}} n_{\alpha,w}(n-\alpha,w)+\sum_{\alpha\in\mathbf{Z}} n_{\alpha,w}(n-\alpha,w+1).$$

Example 2.8. Let X be a smooth space-germ, $Y\subset X$ a reduced divisor, and $x\in Y$. Let $\mathbf{V}$ be a polarized variation of Hodge structure on $X\setminus Y$ such that its underlying representation is abelian and quasi-unipotent. Then one obtains a limit mixed Hodge structure $L\mathbf{V}$ at x equipped with a semi-simple action of $H_1(X\setminus Y)$, cf. [9], i.e., an object of $MHS(H_1(X\setminus Y))$.

If Y has irreducible components $Y_1,\dots,Y_s$, then $H_1(X\setminus Y)$ is free abelian on generators $M_1,\dots,M_s$, where M_j is represented by an oriented circle in a transverse slice to Y_j.

Lemma 2.9. a) *There is an unique way to extend the definition of $Spp_\psi(\mathcal{M},g,x)$ to $\mathcal{M}\in Ob\,D^bMHM(X)$ in such a way that for any distinguished triangle*

$$\mathcal{M}'\to\mathcal{M}\to\mathcal{M}''\xrightarrow{+1}$$

one has

$$Spp_\psi(\mathcal{M},g,x)=Spp_\psi(\mathcal{M}',g,x)+Spp_\psi(\mathcal{M}'',g,x).$$

b) *For $u\in\mathcal{O}^*_{X,x}$ one has*

$$Spp_\psi(\mathcal{M},ug,x)=Spp_\psi(\mathcal{M},g,x).$$

c) *$Spp_\psi(\mathcal{M},g,x)=\sum_l Spp_\psi(Gr^W_l\mathcal{M},g,x)$ for $\mathcal{M}\in Ob\,MHM(X)$.*

d) *Let $T(p,q):\mathbf{Z}[\mathbf{Q}\times\mathbf{Z}]\to\mathbf{Z}[\mathbf{Q}\times\mathbf{Z}]$ be the automorphism mapping (α,w) to $(\alpha+p,w+p+q)$. Then*

$$Spp_\psi(\mathcal{M}\otimes\mathbf{Q}^H_X(k),g,x)=T(-k,-k)(Spp_\psi(\mathcal{M},g,x)).$$

e) *We let $\mathcal{H}^ji^*_x\mathcal{M}\in MHS(\mathbf{Z})$ with trivial representation. Then*

$$Spp_\varphi(\mathcal{M},g,x)=Spp_\psi(\mathcal{M},g,x)+\sum_j(-1)^jSpp(\mathcal{H}^ji^*_x\mathcal{M}).$$

f) *Let $c_n:\mathbf{Z}[\mathbf{Q}\times\mathbf{Z}]\to\mathbf{Z}[\mathbf{Q}\times\mathbf{Z}]$ $(n\in\mathbf{N}^*)$ be the unique map which sends (α,w) to $\sum_{k=0}^{n-1}([\alpha]+\frac{\{\alpha\}+k}{n},w)$, (here $[\beta]$ (resp. $\{\beta\}=\beta-[\beta]$) is the integer part (resp. the fractional part) of β). Then:*

$$Spp_\psi(\mathcal{M},f^n,x)=c_n\,Spp_\psi(\mathcal{M},f,x).$$

The properties a)–d) *also hold with φ instead of ψ.*

The proof is left to the reader.

During the paper the notation $\mathbf{Q}^H_X$ means $a^*_X\mathbf{Q}^H_{pt}$, where $a_X:X\to pt$ is the constant function (see [10], p. 324).

3. The General Setup

Assume a complex analytic space X and a point $x \in X$ are given. The invariant $Spp_\psi(\mathcal{M}, f, x)$ depends on $\mathcal{M}$ and on f in a complicated way. We want to decompose it into one step depending only on $\mathcal{M}$, and a combinatorial step depending mainly on f. To illustrate this, we first treat the case where $\dim(X) = 1$.

Example 3.1. Let X be one-dimensional, $x \in X$ and $f : X \to \mathbf{C}$ non-constant holomorphic with $f(x) = 0$. Assume that X is irreducible at x. Let $\mathcal{M}$ be a mixed Hodge module on X. We will indicate how to compute $Spp_\psi(\mathcal{M}, f, x)$.

Let $\sigma : \tilde{X} \to X$ be the normalization of X, and let t be a uniformizing parameter at $\tilde{x} = \sigma^{-1}(x)$. Then $f \circ \sigma = u \cdot t^n$ for some germ $u \in \mathcal{O}^*_{\tilde{X},\tilde{x}}$ and $n \in \mathbf{N}^*$. If $\mathcal{N} = \mathcal{H}^0\sigma^*\mathcal{M}$, then by (4) and Lemma 2.9 b and f, one has:

$$Spp_\psi(\mathcal{M}, f, x) = c_n\ Spp_\psi(\mathcal{N}, t, \tilde{x}).$$

Moreover, $Spp_\psi(\mathcal{N}, t, \tilde{x}) = Spp(L\mathcal{M}, T_s)$ with $L\mathcal{M}$ the limit mixed Hodge structure of $\mathcal{N}$ at $\tilde{x}$ (observe that the restriction of $\mathcal{N}$ to a punctured neighbourhood of $\tilde{x}$ is an admissible variation of mixed Hodge structure) and T_s is the semi-simple part of the monodromy T. Hence $Spp_\psi(\mathcal{M}, f, x) = c_n\ Spp(L\mathcal{M})$. Here $L\mathcal{M}$ depends only on $\mathcal{M}$, and n depends only on f. This means that the computation of $Spp_\psi(\mathcal{M}, f, x)$ goes in two steps. The first one, the computation of $L\mathcal{M}$ as an object of $MHM(\mathbf{Z})$, does not involve f. In the second step, only the multiplicity n of $\tilde{x}$ as a zero of $f \circ \sigma$ matters.

We are going to generalize the previous example to the two-dimensional case. The first step, passage from $\mathcal{M}$ to $L\mathcal{M}$, is possible if $\mathcal{M}$ has an abelian monodromy group, which satisfies the condition of Example 2.8, and gives rise to an object $L\mathcal{M}$ of $MHM(G)$, where $G = H_1(X \setminus Y)$ and Y is the critical locus of $\mathcal{M}$. The second step involves identification of the relevant discrete invariants of the function $f : X \to \mathbf{C}$ at x. We will use the decorated resolution graph Γ of f with respect to Y, to be defined in Example 4.5. We will also define a map (see Definition 4.4)

$$Spp_\Gamma : MHM(G) \to \mathbf{Z}[\mathbf{Q} \times \mathbf{Z}]$$

with the property that

$$Spp_\psi(\mathcal{M}, f, x) = Spp_\Gamma(L\mathcal{M}),$$

provided that $\mathbf{V} = j^*\mathcal{M}$ is a polarized variation of Hodge structure, and $\mathcal{M} = j_*j^*\mathcal{M}$, where $j : X \setminus Y \to X$ is the inclusion.

This is the main result of the paper.

4. The Definition of Spp_Γ

In this section X is a two-dimensional analytic space, $Y \subset X$ is a reduced divisor, $x \in Y$ a normal singularity of X. Assume that (X, Y) is contractible onto x. Let $\mathcal{S}(Y)$ be the set of irreducible components of Y at x.

Definition 4.1. A *decorated graph* for (Y, x) is a finite connected graph Γ, without edges connecting a vertex to itself, with set of vertices $\mathcal{V}$ and set of edges $\mathcal{E}$ and the following data and conditions:

a) $\mathcal{V} = \mathcal{D} \sqcup \mathcal{S}$ with $\mathcal{D}$, $\mathcal{S}$ non-empty and an injection $\mathcal{S}(Y) \hookrightarrow \mathcal{S}$;

b) a map $e : \mathcal{D} \to \mathbf{Z}$ such that the matrix A on $\mathcal{D} \times \mathcal{D}$ given by

$$A(d,d') = \begin{cases} e(d) & \text{if } d = d' \ ; \\ 0 & \text{if } d \neq d' \text{ and } (d,d') \notin \mathcal{E}; \\ 1 & \text{if } d \neq d' \text{ and } (d,d') \in \mathcal{E} \end{cases}$$

is negative definite;

c) a map $g : \mathcal{D} \to \mathbf{N}$;

d) a map $m : \mathcal{S} \to \mathbf{N}$ taking at least one non-zero value.

e) For any $d \in \mathcal{D}$, let $\mathcal{V}_d = \{v \in \mathcal{V} \,|\, \mathrm{dist}(v,d) = 1\}$ be the set of neighbors of d in Γ. Let $\mathbf{Z}^{\mathcal{V}}$ be the free abelian group generated by $\{[v]\}_{v\in\mathcal{V}}$. Define the group $G(\Gamma)$ as the quotient of $\mathbf{Z}^{\mathcal{V}}$ by the subgroup generated by the following relations:

$$e(d)[d] + \sum_{v\in\mathcal{V}_d} [v] = 0 \quad (d \in \mathcal{D})$$

or

$$\sum_{d'\in\mathcal{D}} A(d,d')[d'] + \sum_{v\in\mathcal{V}_d\setminus\mathcal{D}} [v] = 0 \quad (d \in \mathcal{D}).$$

Let l be the composition $\mathbf{Z}^{\mathcal{S}} \hookrightarrow \mathbf{Z}^{\mathcal{V}} \to G(\Gamma)$, and let $m : \mathbf{Z}^{\mathcal{S}} \to \mathbf{Z}$ be the linear extension of m (i.e., $m[s] = m(s)$). Then we assume that m can be extended to $G(\Gamma)$, i.e., there exists $m_0 : G(\Gamma) \to \mathbf{Z}$ such that $m_0 \circ l = m$.

Our maps are summarized in the following diagram:

$$\begin{tikzcd}
 & & & \mathrm{coker}(A) \\
 & \mathbf{Z}^{\mathcal{V}} \arrow[r, two heads] & G(\Gamma) \arrow[ur] \arrow[d, dashed] & \\
 & \mathbf{Z}^{\mathcal{S}} \arrow[u] \arrow[ur, "l"] \arrow[r, "m"] & \mathbf{Z} & \\
0 \arrow[ur] & & &
\end{tikzcd}$$

Notice that coker$A$ is a finite group of order $\det A$, therefore if A is unimodular l is an isomorphism and the assumption in e) is automatically satisfied.

Definition 4.2. For $v \in \mathcal{V}$ we define: $m_v = m_0([v])$, $\delta_v = \#\mathcal{V}_v$, and $M_v \in G(\Gamma)$ as the image of $[v]$ by the natural projection. For $d \in \mathcal{D}$ we denote $g_d := g(d)$.

It is a well-known fact that all the entries of the matrix $-A^{-1}$ are strictly positive if A is a matrix as in Definition 4.1.b. In particular, $m_d > 0$ for any $d \in \mathcal{D}$.

Definition 4.3. Fix a character $\chi : G(\Gamma) \to \mathbf{C}^*$ of finite order. Let $\beta_v \in [0,1)$ be such that $\exp 2\pi i \beta_v = \chi([v])$.

For $d \in \mathcal{D}$ we define $Spp_d(\chi)$ as follows.

For each $v \in \mathcal{V}_d$ and $k \in \{0, \dots, m_d - 1\}$ define:

$$R_d^{kv} = \{-\beta_v + \frac{m_v}{m_d}(k + \beta_d)\};$$

$$\alpha_d^k = \{\frac{k + \beta_d}{m_d}\}.$$

We let $R_d^k = \sum_{v \in \mathcal{V}_d} R_d^{kv}$, and $\delta_d^k = \#\{v : \ R_d^{kv} \neq 0\}$. Then $Spp_d^k(\chi) :=$

$$\begin{cases} -(\alpha_d^k, 0) + (\delta_d - 1) \cdot (\alpha_d^k + 1, 2) + g_d(\alpha_d^k, 1) + g_d(\alpha_d^k + 1, 1) & \text{if } R_d^k = 0 \\ (g_d + R_d^k - 1)(\alpha_d^k, 1) + (g_d + \delta_d^k - R_d^k - 1)(\alpha_d^k + 1, 1) + (\delta_d - \delta_d^k)(\alpha_d^k + 1, 2) & \\ & \text{else,} \end{cases}$$

and

$$Spp_d(\chi) := \sum_{k=0}^{m_d - 1} Spp_d^k(\chi).$$

For $e = (v, w) \in \mathcal{E}$ we define $Spp_e(\chi)$ as follows. Let $m_e := g.c.d.(m_v, m_w)$. The system of equations:

$$\begin{cases} \{\beta_v\} = \{m_v \gamma_e / m_e\} \\ \{\beta_w\} = \{m_w \gamma_e / m_e\} \end{cases}$$

either has a solution $\gamma_e \in \mathbf{R}/\mathbf{Z}$ or has not. We define $Spp_e(\chi)$ by:

$$Spp_e(\chi) := \begin{cases} \sum_{k=0}^{m_e - 1} (\{\frac{k+\gamma_e}{m_e}\}, 0) - (\{\frac{k+\gamma_e}{m_e}\} + 1, 2) & \text{if } \gamma_e \text{ exists} \\ 0 & \text{otherwise} \end{cases}$$

Finally, we let

$$Spp_\Gamma(\chi) := \sum_{d \in \mathcal{D}} Spp_d(\chi) + \sum_{e \in \tilde{\mathcal{E}}} Spp_e(\chi),$$

where $\tilde{\mathcal{E}} := \mathcal{E} \cap (\mathcal{D} \times \mathcal{D})$.

$R_d^k \in \mathbf{N}$, therefore $Spp_\Gamma(\chi) \in \mathbf{Z}[\mathbf{Q} \times \mathbf{Z}]$.

Definition 4.4. Let $\rho \in MHS(G(\Gamma))$. The representation ρ^{pq} splits into a direct sum of characters

$$\rho^{pq} = \oplus_{i=1}^{d(p,q)} \chi_i^{pq}, \quad d(p,q) = \dim H^{p,q}.$$

We define

$$Spp_\Gamma(\rho) := \sum_{p,q} \sum_{i=1}^{d(p,q)} T(p,q) Spp_\Gamma(\chi_i^{pq}).$$

Example 4.5. Let X be a two-dimensional complex analytic space with normal singularity at x. Let $Y \subset X$ be a (reduced) divisor such that the pair (X, Y) is contractible to x and $X \setminus Y$ is smooth. As in Section 4, $\mathcal{S}(Y)$ is the set of irreducible components of Y at x.

We now consider a holomorphic function $p : X \to \mathbf{C}$ and construct a decorated graph Γ, the decorated resolution graph of p with respect to (Y, x). The point $x \in X$ is an isolated singular point of the reduced curve $p^{-1}(0) \cup Y$. We let $\mathcal{S}$ denote the set of branches of $p^{-1}(0) \cup Y$ at x; then $\mathcal{S}(Y) \subset \mathcal{S}$. Let $\pi : U \to X$ be an embedded good resolution of $p^{-1}(0) \cup Y$. Then $D := \pi^{-1}(Y \cup p^{-1}(0))$ is a union of smooth curves on the two-dimensional complex manifold U. Let $E = \pi^{-1}(x)$

and let $\mathcal{D}$ be the set of irreducible components of E, and $\mathcal{V} = \mathcal{D} \sqcup \mathcal{S}$. We assume that $\mathcal{D} \neq \emptyset$. For $v \in \mathcal{V}$ we let D_v be the corresponding irreducible component of E if $v \in \mathcal{D}$ and the strict transform of the corresponding local irreducible component of $Y \cup p^{-1}(0)$ for $v \in \mathcal{S}$. The edges of Γ are pairs (v, w) for which $v \neq w$ and $E \cap D_v \cap D_w \neq \emptyset$. We let $g(d) =$ the genus of D_d and $e(d) = D_d \cdot D_d$ for $d \in \mathcal{D}$. The matrix A as defined in (4.1.b) is then the intersection matrix of the components of E, which is negative definite. Finally we let $m(v)$ be the order of zero of p along D_v for $v \in \mathcal{S}$, or even for $v \in \mathcal{V}$. Then m vanishes on the relations (4.1.e) because the divisor $\pi^*(p^{-1}(0))$ on U is linearly equivalent to zero, hence has zero intersection product with each D_d $(d \in \mathcal{D})$. The induced map with source $G(\Gamma)$ is m_0.

Each M_v $(v \in \mathcal{V})$ can be represented in $H_1(X \setminus p^{-1}(0) \cup Y)$ by an oriented circle in a transversal slice to D_v. They generate the subgroup $G(\Gamma)$ of $H_1(X \setminus p^{-1}(0) \cup Y)$. Actually, there exists an exact sequence

$$0 \to G(\Gamma) \xrightarrow{i} H_1(X \setminus p^{-1}(0) \cup Y) \to H_1(E) \to 0.$$

Since $H_1(E)$ is a torsion free group, the above sequence splits.

Notice that for any $s \in \mathcal{S}$ we have exactly one $d_s \in \mathcal{D}$ such that $(s, d_s) \in \mathcal{E}$.

5. The Main Result

Assumption: In this section, X is a two-dimensional complex analytic space, $x \in X$ a normal point on X, and $Y \subset X$ a reduced divisor with $x \in Y$. Assume that $X \setminus Y$ is smooth and connected. Let $\mathbf{V}$ be a polarized variation of Hodge structure on $X \setminus Y$ such that its underlying representation ρ is abelian and quasi-unipotent. Consider $K := im\,\rho \subset Aut(H)$ and its torsion subgroup T. If $K/T \neq 0$, we assume that there exist $w_j \in \mathcal{O}_X$ $(j = 1, \ldots, s)$, such that $Y = \cup_{j=1}^{s} Z(w_j)$, and $w = (w_1, \ldots, w_s) : X \setminus Y \to (\mathbf{C}^*)^s$ induces an epimorphism $w_* : H_1(X \setminus Y) \to \mathbf{Z}^s$ which fits in the following commutative diagram:

$$\begin{tikzcd} \pi_1(X \setminus Y) \arrow[r, "\rho"] \arrow[d, "w_*"] & K \arrow[d] \\ \mathbf{Z}^s \arrow[r, "\rho_f"] & K/T \end{tikzcd}$$

For such a $\mathbf{V}$, the limit mixed Hodge structure $L\mathbf{V} \in MHS(H_1(X \setminus Y))$ exists by [9] (cf. Example 2.8). Let $\mathcal{M} = j_*\mathbf{V}$ where $j : X \setminus Y \to X$ is the natural inclusion.

Let $p : X \to \mathbf{C}$ be a holomorphic function. Let Γ be a decorated resolution graph of p with respect to (Y, x) (cf. Example 4.5) and $Spp_\Gamma(L\mathbf{V})$ the invariant defined in Definition 4.4 via the composed map $G(\Gamma) \hookrightarrow H_1(X \setminus p^{-1}(0) \cup Y) \to H_1(X \setminus Y)$. Our key result is:

Theorem 5.1. *Let X and $\mathcal{M}$ be as above. Then*:

$$Spp_\psi(\mathcal{M}, p, x) = Spp_\Gamma(L\mathbf{V}).$$

Recall that the spectrum $Spp_\psi(\mathcal{M},p,x)$ of a mixed Hodge module $\mathcal{M}$ with irreducible one-dimensional support Y is zero if $p|Y \equiv 0$; otherwise it can be computed as follows. Since $\mathcal{M}$ is a polarizable admissible variation of Hodge structure on $Y \setminus \{x\}$, it has a limit mixed Hodge structure $L\mathcal{M}$. The topological information from p is the degree $\deg(p|Y)$ of the map $p|Y : Y \to \mathbf{C}$. Then $Spp_\psi(\mathcal{M},p,x) = c_{\deg(p|Y)}(Spp(L\mathcal{M}))$ (cf. Example 3.1).

If Y_s is one of the irreducible components of the critical locus Y of $\mathcal{M}$ and Γ is a decorated resolution graph of p with respect to (Y,x), then $\deg(p|Y_s) = m(d_s)$ where $(s,d_s) \in \mathcal{E}$.

Theorem 5.2. *Let X and $\mathcal{M}$ be as in the Assumption. Let $Y = \cup_{s\in\mathcal{S}(Y)} Y_s$ be the irreducible decomposition of (Y,x), $i_{Y_v} : Y_v \to X$ and $j : X \setminus Y \to X$ the natural inclusions. Then*:

$$Spp_\psi(\mathcal{M},p,0) = Spp_\Gamma(Lj^*\mathcal{M}) + \sum_{\substack{s\in S(Y)\\ p|Y_s\not\equiv 0}} \sum_k (-1)^k c_{m(d_s)}\ Spp(L\mathcal{H}^k i^!_{Y_s}\mathcal{M}).$$

Proof. Use (2), Lemma 2.9, and Theorem 5.1. ☐

5.1. The proof of Theorem 5.1. The proof is divided into three steps.

Step 1. Let $\pi : U \to X$ be a resolution of $p^{-1}(0) \cup Y$ as in Example 4.5.

Set $\mathcal{N} = \mathcal{H}^0\pi^*\mathcal{M}$ and $\mathcal{N}_E = i_E^*\psi_{p\circ\pi}\mathcal{N}$, where $i_E : E = \pi^{-1}(0) \to U$ is the natural inclusion. Now it is clear that $\mathcal{H}^0\pi_*\mathcal{N} = \mathcal{M}$ modulo terms with support in $\{0\}$, and supp $\mathcal{H}^j\pi_*\mathcal{N} \subset \{0\}$ if $j \neq 0$. Therefore, $\psi_p\mathcal{H}^j\pi_*\mathcal{N} = \psi_p\mathcal{M}$ if $j = 0$ and $= 0$ if $j \neq 0$. Hence, by (4),

$$\mathcal{H}^j\pi_*\psi_{p\circ\pi}\mathcal{N} = 0 \ \text{ if } \ j \neq 0. \tag{5}$$

Consider the following isomorphisms: $i_0^*\psi_p\mathcal{M} = i_0^*\psi_p\mathcal{H}^0\pi_*\mathcal{N} \overset{(4)}{=} i_0^*\mathcal{H}^0\pi_*\psi_{p\circ\pi}\mathcal{N} \overset{(5)}{=} i_0^*\pi_*\psi_{p\circ\pi}\mathcal{N} \overset{(*)}{=} \pi_*i_E^*\psi_{p\circ\pi}\mathcal{N} = \pi_*\mathcal{N}_E$.

The relation $(*)$ follows from $i_0^*\pi_* = \pi_*i_E^*$. For this, notice, that $\pi_* = \pi_!$ because π is proper ([10] (4.3.3)), and then use [loc. cit.] (4.4.3).

For each $d \in \mathcal{D}$, let $\tilde{D}_d = D_d \setminus \cup_{d'\in\mathcal{D}\cap\mathcal{V}_d} D_{d'}$ and $k_d : \tilde{D}_d \hookrightarrow D_d$ its inclusion. For each $e = (d,d') \in \tilde{\mathcal{E}}$ denote by $i_e : D_d \cap D_{d'} \to E$ the natural inclusion. Then by (2) one has the following distinguished triangle:

$$\to \oplus_{e\in\tilde{\mathcal{E}}}(i_e)_*i_e^!\mathcal{N}_E \to \mathcal{N}_E \to \oplus_{d\in\mathcal{D}}(k_d)_*k_d^*\mathcal{N}_E \to$$

By the additivity of the functor Spp, we obtain:

$$Spp(i_0^*\psi_p\mathcal{M}) = Spp(\pi_*\mathcal{N}_E) = \sum_{e\in\tilde{\mathcal{E}}} Spp(i_e^!\mathcal{N}_E) + \sum_{d\in\mathcal{D}} Spp(\pi_*(k_d)_*k_d^*\mathcal{N}_E). \tag{6}$$

(Everywhere, the action is the natural monodromy provided by ψ.)

Fix $d \in \mathcal{D}$. Let $D'_d = \tilde{D}_d \setminus St(p^{-1}(0) \cup Y) = D_d \setminus \cup_{v\in\mathcal{V}_d} D_v$, and $j_d : D'_d \hookrightarrow D_d$ denotes the natural inclusion.

Lemma 5.3. *The restriction to D'_d of the module $\mathcal{N}'_E = (k_d)_*k_d^*\mathcal{N}_E$ is smooth (i.e., rat $j_d^*\mathcal{N}'_E$ is a local system). Moreover, it satisfies*

$$\mathcal{N}'_E = (j_d)_*(j_d)^*\mathcal{N}'_E. \tag{7}$$

Proof. By construction: $j_d^*\mathcal{N}'_E = j_d^*\mathcal{N}_E = j_d^*\psi_{p\circ\pi}\mathcal{N}$.

Since $rat\mathcal{N}$ restricted to $U \setminus D$ is a local system, and D'_d is a smooth divisor in $U \setminus \cup_{v\in\mathcal{V}_d} D_v$, the sheaf $rat\, j_d^*\psi_{p\circ\pi}\mathcal{N}$ is a local system, too.

The obstruction to the isomorphism (7) lies in the points $P \in D_d \cap (\cup_{v\in\mathcal{S}} D_v) = \tilde{D}_d \setminus D'_d$.

Take $P = D_d \cap D_v$ such that $p|Y_v \not\equiv 0$ $(v \in \mathcal{S})$. Then the assumption $\mathcal{M} = j_*j^*\mathcal{M}$ and (2) give $\psi_{p\circ\pi} i^!_{D_v}\mathcal{N} = 0$. Now (3) and the commutativity of the vanishing cycle functors complete the argument in this case.

If $P = D_d \cap D_v$ such that $v \in \mathcal{S}$ and $p|Y_v \equiv 0$, then see [11] (4.7). $\square$

Notice that $j_d^*\mathcal{N}'_E = j_d^*\mathcal{N}_E$, and by Lemma 5.3, $(k_d)_*k_d^*\mathcal{N}_E = (j_d)_*j_d^*\mathcal{N}_E$. Since the isomorphism $\mathbf{H}^\bullet(D_d, (j_d)_*j_d^*\mathcal{N}_E) = \mathbf{H}^\bullet(D'_d, j_d^*\mathcal{N}_E)$ is compatible with the mixed Hodge structures, one has:

$$Spp(\pi_*(k_d)_*k_d^*\mathcal{N}_E) = Spp(\mathbf{H}^\bullet(D'_d, j_d^*\mathcal{N}_E)).$$

STEP 2. *The identity* $Spp(\mathbf{H}^\bullet(D'_d, j_d^*\mathcal{N}_E)) = Spp_d(L\mathbf{V})$.

By the additivity of Spp (see 2.13.a) it is enough to prove

$$\sum_l Spp(\mathbf{H}^\bullet(D'_d, Gr_l^W j_d^*\mathcal{N}_E)) = Spp_d(L\mathbf{V}). \tag{8}$$

Since $j_d^*\mathcal{N}_E$ is smooth (Lemma 5.3), $\mathbf{V}_{d,l} := Gr_l^W j_d^*\mathcal{N}_E$ is a polarizable variation of Hodge structure on D'_d.

Lemma 5.4. *The representation associated with the local system* $rat\mathbf{V}_{d,l}$ *is abelian and quasi-unipotent.*

Proof. D'_d has a neighbourhood homeomorphic to $D'_d \times \{disc\}$. $\square$

The importance of this lemma appears in the following:

Lemma 5.5. *Any polarizable variation of Hodge structure on a quasi-projective smooth curve* $\mathcal{C}$*, whose underlying local system has a monodromy representation which is abelian and quasi-unipotent, is locally constant.*

Proof. Since the monodromy representation on $\mathcal{C}$ is semi-simple ([1], (4.2.6)), it follows, that it is a direct sum of one-dimensional representations, which are finite. Hence a finite cover $\tilde{\mathcal{C}}$ of $\mathcal{C}$ has trivial local and global monodromies. Therefore a global marking $\tilde{\mathcal{C}} \to \mathbf{D}$ can be defined in the moduli space of Hodge structures. This by Griffiths' theorem [4] can be extended to the smooth closure of $\tilde{\mathcal{C}}$. But this extended map is trivial by the rigidity theorem [13]. $\square$

Consider now the limit mixed Hodge structure $L\mathbf{V} \in MHS(H_1(X \setminus Y))$. Its representation defines a locally constant abelian variation $\mathcal{M}_\infty$ on $X \setminus Y$. In [9], among other facts, the following is proved:

Lemma 5.6. *Let* $(C, x) \subset (X, x)$ *be a curve with* $C \cap Y = \{x\}$. *Let* $L(\mathcal{M}|C)$*, respectively* $L(\mathcal{M}_\infty|C)$ *be the limit mixed Hodge structures at* x *of the restrictions of* $\mathcal{M}$*, respectively of* $\mathcal{M}_\infty$*, to* C. *Then* $Gr^W L(\mathcal{M}|C) = Gr^W L(\mathcal{M}_\infty|C)$.

Now, if we replace in the above construction $\mathcal{M}$ by $\mathcal{M}_\infty$, then we obtain a variation $\mathbf{V}_{\infty,d,l}$ instead of $\mathbf{V}_{d,l}$

Lemma 5.7. *The variations of Hodge structure $\mathbf{V}_{d,l}$ and $\mathbf{V}_{\infty,d,l}$ on D'_d are isomorphic.*

Proof. Both are abelian variations by Lemma 5.4, with flat Hodge bundles by Lemma 5.5. By construction, the underlying representations are the same. We have only to show that in a fixed point $P \in D'_d$ the stalks are isomorphic (by an isomorphism, which is compatible with the representations).

Let C be a transversal slice to D'_d at a point $P \in D'_d$ and t a uniformizing parameter of (C,P). Then $(\mathbf{V}_{d,l})_P \simeq Gr^W_l \psi_{t^{m_d}}(\mathcal{M}|C) \simeq \oplus_{i=1}^{m_d} Gr^W_l L(\mathcal{M}|C)$. Similar isomorphisms holds for the other variation, therefore the result follows from Lemma 5.6. The compatibility follows from the naturality of the constructions. $\square$

Therefore, (8) is equivalent to

$$\sum_l Spp(\mathbf{H}^\bullet(D'_d,\mathbf{V}_{\infty,d,l})[1]) = Spp_d(L\mathbf{V}). \tag{9}$$

Notice that both sides of (9) depend only on the limit mixed Hodge structure $L\mathbf{V}$.

Now, $(L\mathbf{V},\rho) \in MHS(H_1(X \setminus Y))$ splits in a direct sum $\rho = \oplus_{p,q} \oplus_{i=1}^{d(p,q)} \chi_i^{p,q}$, $d(p,q) = \dim L\mathbf{V}^{p,q}$. The construction of $\mathbf{V}_{\infty,d,l}$ preserves this decomposition, therefore $\mathbf{V}_{\infty,d,l} = \oplus_{p+q=l} \oplus_{i=1}^{d(p,q)} \mathbf{V}^{p,q,i}_{\infty,d}$. We have to show that

$$Spp(\mathbf{H}^\bullet(D'_d,\mathbf{V}^{p,q,i}_{\infty,d})[1]) = Spp_d(\chi_i^{p,q}). \tag{10}$$

In the sequel we omit the indices p,q and i. Moreover, we can assume that χ is of type $(p,q)=(0,0)$.

Lemma 5.8. *The variation $\mathbf{V}_{\infty,d}$ is m_d-dimensional. It has a direct sum decomposition $\oplus_{k=0}^{m_d-1}\mathbf{V}^k_d$ in one-dimensional locally constant variations of $\mathbf{C}$-Hodge structure (of the same type $(0,0)$), such that the monodromy of $\mathbf{V}^k_d$ around the points $(P_d \cap P_v)_{v\in\mathcal{V}_d}$ is $\exp(-2\pi i R^{kv}_d)$. The monodromy action on $\mathbf{V}^k_d$ given by the vanishing cycle functor is $\exp(2\pi i \alpha^k_d)$.*

Proof. The verification is local in small neighbourhoods of the points $D_d \cap D_v$ ($v \in \mathcal{V}_d$). Here, in a suitable coordinate system $\rho\circ\pi = x^{m_d}y^{m_v}$. The verification is left to the reader. $\square$

Proof of (10). By Lemma 5.8, we have to verify only:

$$Spp(\mathbf{H}^\bullet(D'_d,\mathbf{V}^k_d)[1]) = Spp^k_d(\chi). \tag{11}$$

In order to compute the left hand side of (11), we have to compute the dimensions h^{pq} of the spaces $Gr^W_{p+q}Gr^p_F\mathbf{H}^\bullet(D'_d,\mathbf{V}^k_d)$. Then $h^{pq}_\lambda = h^{pq}$ if $\lambda = \exp(2\pi i\alpha^k_d)$, and $=0$ otherwise.

Let $\Omega^\bullet(\log\Sigma_d)$ be the complex of meromorphic differentials on D_d with at worst logarithmic poles along $\Sigma_d = \cup_{v\in\mathcal{V}_d}(D_d\cap D_v)$, and let $\mathcal{V}$ denote Deligne's canonical extension of $\mathbf{V}^k_d \otimes_{\mathbf{C}} \mathcal{O}_{D'_d}$. Now, $H^*(D'_d,\mathbf{V}^k_d) = \mathbf{H}^*(D_d,K^\bullet)$, where $K^\bullet$ is the complex $\{\nabla : \mathcal{V} \to \Omega^1(\log\Sigma_d)\otimes\mathcal{V}\}$. The Hodge filtration of this complex is Deligne's "filtration bête" $\sigma_{\geq p}$, therefore the first term of the Hodge spectral sequence is ${}_F E_1^{pq} = H^q(D_d,K^p)$. This spectral sequence degenerates at E_1.

If $R^k_d = 0$, then $\mathcal{V} = \mathcal{O}_{D'_d}$. Hence $E_1^{00} = \mathbf{C}$, $E_1^{01} = \mathbf{C}^{g_d}$, and $E_1^{10} = \mathbf{C}^{g_d+\delta_d-1}$. This case corresponds to the trivial flat bundle, therefore we recover exactly the mixed Hodge structure of $D_d \setminus \{\delta_d \text{ points}\}$. In particular, $H^0 = \mathbf{C}$ has type $(0,0)$,

E_1^{01} has type $(0,1)$, $\mathbf{C}^{g_d} \subset E_1^{10}$ has type $(1,0)$, and $\mathbf{C}^{\delta_d - 1} = E_1^{10}/\mathbf{C}^{g_d}$ has type $(1,1)$.

Assume that $R_d^k \neq 0$. Then ${}_F E_1^{pq} = 0$ if $p+q \neq 1$ because $\deg \mathcal{V} = -R_d^k < 0$ and $\deg(\mathcal{O}(-\Sigma) \otimes \mathcal{V}^*) = R_d^k - \delta_d < 0$. Then by the Riemann-Roch Theorem, $\dim E_1^{01} = g_d + R_d^k - 1$, and $\dim E_1^{10} = g_d + \delta_d - R_d^k - 1$. This means that the only nontrivial cohomological group is $H^1 = H^1(D'_d, \mathbf{V}_d^k)$. In order to compute its Hodge numbers h^{pq}, we need the weight filtration too.

The weight filtration of the complex $K^\bullet$ is $W_{-1}K^\bullet = 0$, $W_0 K^\bullet = \{\nabla : \mathcal{V} \to im\, \nabla\}$, and $W_1 K^\bullet = K^\bullet$. The extension W_0 is a resolution of $j_* \mathbf{V}_d^k$ and (by [20]) $H^1(D_d, j_* rat \mathbf{V}_d^k)$ is pure of weight 1. On the other hand $R^1 j_* rat \mathbf{V}_d^k = \mathbf{C}^{\delta_d - \delta_d^k}$ and it induces in H^1 a quotient of weight 2. Therefore, the only nonzero Hodge numbers are: h^{10}, h^{01} and h^{11}. More precisely: $h^{11} = \delta_d - \delta_d^k$, $h^{01} = g_d + R_d^k - 1$, and $h^{10} = g_d + \delta_d^k - R_d^k - 1$. Now the expression for the spectral pairs readily follows. $\square$

Example 5.9. Assume that $\delta_d = 1$. Then $R_d^k = 0$ for any k. Therefore $Spp_d^k = -(\alpha_d^k, 0) + g_d(\alpha_d^k, 1) + g_d(\alpha_d^k + 1, 1)$ for any k.

Example 5.10. Assume that $g_d = 0$, $\delta_d = 2$ and $\mathcal{V}_d = \{v, w\}$. Then $R_d^k = R_d^{kv} + R_d^{kw} \in \{0, 1\}$. Using the obvious fact that for $x, y \in \mathbf{R}$ with $x + y \in \mathbf{Z}$ one has $\{x\} + \{y\} = 1$ if $x \notin \mathbf{Z}$, and $= 0$ if $x \in \mathbf{Z}$, we obtain that $R_d^k = 1$ if $\beta_v - \frac{m_v}{m_d}(k + \beta_d) \notin \mathbf{Z}$ and $= 0$ otherwise.

Lemma 5.11. *Let $l_d = g.c.d.(m_d, m_v) = g.c.d.(m_v, m_w)$. The following assertions are equivalent:*

a) *There exists $k_0 \in \mathbf{Z}$ such that $\beta_v - \frac{m_v}{m_d}(k_0 + \beta_d) \in \mathbf{Z}$.*

b) *There exists $\gamma_d \in \mathbf{R}/\mathbf{Z}$ such that*

$$\left\{ \begin{array}{l} \{\beta_v\} = \{m_v \gamma_d / l_d\} \\ \{\beta_d\} = \{m_d \gamma_d / l_d\}. \end{array} \right.$$

c) *There exists $\gamma_d \in \mathbf{R}/\mathbf{Z}$ such that*

$$\left\{ \begin{array}{l} \{\beta_v\} = \{m_v \gamma_d / l_d\} \\ \{\beta_w\} = \{m_w \gamma_d / l_d\}. \end{array} \right.$$

(*Notice that* b) *or* c) *determines γ_d uniquely.*)

Moreover, if one of these conditions hold, then:

$$\beta_v - \frac{m_v}{m_d}(k + \beta_d) \in \mathbf{Z} \Leftrightarrow \frac{(k - k_0) m_v}{m_d} \in \mathbf{Z}.$$

Proof. Use the relation $[v] + [w] + e_d[d] = 0$ (cf. Definition 4.1.e). $\square$

Therefore, the lemma implies that if $g_d = 0$ and $\delta_d = 2$, then

$$Spp_d(\chi) = \left\{ \begin{array}{ll} \sum_{k=0}^{l_d - 1} -(\{\frac{k+\gamma_d}{l_d}\}, 0) + (\{\frac{k+\gamma_d}{l_d}\} + 1, 2) & \text{if } \gamma_d \text{ exists} \\ 0 & \text{otherwise.} \end{array} \right.$$

We will see later that the expression $Spp_e(\chi)$ $(e \in \mathcal{E}')$ is exactly of this type.

STEP 3. *The computation of* $Spp(i^!_e\mathcal{N}_E)$.

Fix a node $e=(v,w)\in\tilde{\mathcal{E}}$. Let us modify the resolution by a blowing up at the point $P=D_v\cap D_w$. The new resolution is $\pi':U'\to U\xrightarrow{\pi}B$. Define $\mathcal{D}'$ and $\mathcal{E}'$ similarly as in the first case. Then $\mathcal{D}'=\mathcal{D}\sqcup\{D_e\}$, $\mathcal{E}'=(\mathcal{E}\setminus\{P\})\sqcup\{P_v,P_w\}$, where D_e is the new exceptional divisor, and $P_v=D_e\cap D_v$, $P_w=D_e\cap D_w$, where the strict transforms of D_v and D_w are denoted by the same symbols. Set $E'=(\pi')^{-1}(0)$, and consider the modules $\mathcal{N}'=\mathcal{H}^0{\pi'}^*\mathcal{M}$ and $\mathcal{N}'_E=\mathcal{H}^0i^*_{E'}\psi_{p\circ\pi'}\mathcal{N}'$.

Lemma 5.12. *The mixed Hodge structures* $i^!_P\mathcal{N}_E$, $i^!_{P_v}\mathcal{N}'_E$ *and* $i^!_{P_w}\mathcal{N}'_E$ *are isomorphic in a way compatible with their monodromy action.*

Proof. In a neighbourhood of P (resp. of P_v) $\mathcal{N}_E=\psi_{p\circ\pi}\mathcal{N}$ (resp. $\mathcal{N}'_E=\psi_{p\circ\pi'}\mathcal{N}'$). On the other hand, $\mathcal{H}^*(i^!_P\psi_{p\circ\pi}\mathcal{N})=\mathbf{H}^*_c(F_{p\circ\pi},\mathcal{N})$, where $F_{p\circ\pi}$ is the Milnor fiber of $p\circ\pi$ in P. Similarly, $\mathcal{H}^*(i^!_{P_v}\psi_{p\circ\pi'}\mathcal{N}')=\mathbf{H}^*_c(F_{p\circ\pi'},\mathcal{N}')$. But, for a suitable representatives, one has an inclusion of $F_{p\circ\pi'}$ into $F_{p\circ\pi}$ (induced by the blowing up map) which is a homotopy equivalence, and it identifies the sheaves $\mathcal{N}$ and $\mathcal{N}'$. Moreover, it preserves the monodromy action too. Since *rat* is an exact functor, the lemma follows. $\square$

Now, if we write (6) for the resolutions π and π', we obtain that

$$Spp(i^!_{P_v}\mathcal{N}'_E)+Spp(i^!_{P_w}\mathcal{N}'_E)+Spp_{D_e}(\chi')=Spp(i^!_P\mathcal{N}_E).$$

Using Lemma 5.12 one has $Spp(i^!_e\mathcal{N}_E)=-Spp_{D_e}(\chi')$. Now, the result follows from the computation of Example 5.10.

6. Examples

6.1. Abelian coverings. Consider a connected normal surface singularity (X,x) and a covering $\phi:(X,x)\to(\mathbf{C}^2,0)$ which is ramified over Δ. Let $\Delta=\cup_{v=1}^{S(\Delta)}\Delta_v$ be the irreducible decomposition of Δ. Consider a germ: $p:(\mathbf{C}^2,0)\to(\mathbf{C},0)$. We are interested in spectral pairs $Spp_\psi(\mathbf{Q}^H_X[2],f,x)$, where f is the composed map $f=p\circ\phi$. The pair $(\phi,(X,x))$ is uniquely determined by the nonramified covering $X^*=\phi^{-1}(B\setminus\Delta)\to B\setminus\Delta$ [16] (here (B,Δ) is a good representative).

In particular, the mixed Hodge module $\mathcal{M}=\mathcal{H}^0\phi_*\mathbf{Q}^H_X[2]$ on $(\mathbf{C}^2,0)$ is completely determined by the exact sequence

$$1\to\pi_1(X^*)\to\pi_1(B\setminus\Delta,*)\xrightarrow{\tau}G\to1.$$

The critical locus of $\mathcal{M}$ is contained in Δ and the representation of $\mathcal{M}|_{B\setminus\Delta}$ is the induced representation by τ of the regular representation ρ_G of G.

Now, (4) assures, that $\mathcal{H}^\bullet i^*_x\psi_f\mathbf{Q}^H_X[2]=\mathcal{H}^\bullet i^*_0\psi_p\mathcal{M}$, therefore

$$Spp_\psi(\mathbf{Q}^H_X[2],f,x)=Spp_\psi(\mathcal{M},p,0).$$

Suppose, that G is abelian. Then Theorem 5.2 can be applied. The variation $\mathcal{M}|_{B\setminus\Delta}$ is a flat variation of Hodge structure. It can be identified with $(\mathbf{Q}^{|G|},F,W)$, where $Gr^p_F\mathbf{Q}^{|G|}=Gr^W_l\mathbf{Q}^{|G|}=0$ if $p\neq0$ or $l\neq0$. The underlying abelian representation is

$$\rho^{ab}=\rho_G\circ\tau^{ab}:H_1(B\setminus\Delta,\mathbf{Z})\to GL(|G|,\mathbf{Q})$$

where $\tau^{ab}:H_1(B\setminus\Delta,\mathbf{Z})\to G$ is induced by τ. Actually, this is the description of $L\mathcal{M}$, too. In particular $Spp_\Gamma(L\mathcal{M})$ is well-defined.

Let $v \in \mathcal{S}(\Delta)$. The variation $i^!_{\Delta_v}\mathcal{M}$ has the following description: If $j : B \setminus \Delta \to B$ is the natural inclusion, then $R^0 j_* j^* \phi_* \mathbf{C}_X = \phi_* \mathbf{C}_X$ and for any point $P_v \in \Delta_v - \{0\}$ one has $\dim(R^1 j_* j^* \phi_* \mathbf{C}_X)_P = \dim(R^0 j_* j^* \phi_* \mathbf{C}_X)_P = a_v$, where above P_v there lie exactly a_v points of X. The above spaces can be recovered from the representation ρ^{ab} via the expression $\mathbf{C}_v^{|G|} = \ker(\rho^{ab}(M_v) - 1) \subset \mathbf{C}^{|G|}$. Let $d_v \in \mathcal{D}$ be the unique vertex such that $(v, d_v) \in \mathcal{E}$. Then $\mathbf{C}_v^{|G|}$ is a sub-Hodge structure of $\mathbf{C}^{|G|}$ with the automorphism $\rho^{ab}([M_{d_v}])$.

The above discussion shows that $\mathcal{M} \to j_* j^* \mathcal{M}$ is one-to-one. In particular, $i_* \mathcal{H}^k i^!_{\Delta_v} \mathcal{M} = 0$ if $k \neq 1$. The exact sequence (3) shows that $\mathcal{H}^1 i^!_{\Delta_v} \mathcal{M}$ is of type $(1,1)$. Its restriction to $\Delta_v - \{0\}$ is a locally constant variation; it can be identified with $\mathbf{C}_v^{|G|}(-1)$ with monodromy $\rho^{ab}([M_{d_v}])$. Similarly as above, its limit $L\mathcal{H}^1 i^!_{\Delta_v}\mathcal{M}$ has the same presentation too.

Finally, notice that degree$(f|\Delta_v) = m(d_v)$.

Proposition 6.1.

$$Spp_\psi(\mathbf{Q}_X^H[2], f, x) = Spp_\Gamma(\mathbf{C}^{|G|}, \rho^{ab}) - \sum_{\substack{v \in \mathcal{S}(\Delta) \\ p|\Delta_v \not\equiv 0}} c_{m(d_v)} Spp(\mathbf{C}_v^{|G|}(-1), \rho^{ab}([M_{d_v}])).$$

Example 6.2. The case of Hirzebruch-Jung singularities. Let (z_1, z_2) be a local coordinate system in $(\mathbf{C}^2, 0)$. We make the above formula more explicit in the case when $\Delta = \{z_1 z_2 = 0\}$.

Let $A_{n,q} = \mathbf{C}^2/\mathbf{Z}_n$ be the cyclic quotient singularity, where the action of $G = \mathbf{Z}_n$ is given by $(u_1, u_2) \mapsto (\zeta u_1, \zeta^p u_2)$, $\zeta = \exp(2\pi i/n)$ and $pq \equiv 1 (mod\, n)$. Then $\phi : A_{n,q} \to \mathbf{C}^2$ given by $z_i = u_i^n$ $(i = 1, 2)$ defines a covering which is ramified over $\{z_1 z_2 = 0\}$.

In this case, the covering transformation group is $G = \mathbf{Z}_n$, and

$$\tau^{ab} : H_1(B \setminus \Delta, \mathbf{Z}) = \mathbf{Z}^2 \to \mathbf{Z}_n$$

is $\tau^{ab}(e_1) = \hat{1}$, $\tau^{ab}(e_2) = \widehat{-q}$.

On the other hand, for any $v \in \mathcal{S}(\Delta)$ one has $\mathbf{C}_v^{|G|} = \mathbf{C}$ and the transformation $\rho^{ab}([M_{d_v}])$ is the identity. Therefore:

$$Spp_\psi(\mathbf{Q}_X^H[2], p \circ \phi, x) = Spp_\Gamma(\mathbf{C}^n, \rho^{ab}) - \sum_{\substack{v \in \mathcal{S}(\Delta) \\ p|\Delta_v \not\equiv 0}} \sum_{k=0}^{m_{d_v}-1} \left(1 + \frac{k}{m_{d_v}}, 2\right).$$

Example 6.3. Take $p(z_1, z_2) = z_1 + z_2$ in Example 6.2. For $x \in \mathbf{R}$ take $\delta(x) = 0$ if $x \in \mathbf{Z}$, and $= 1$ if $x \notin \mathbf{Z}$. Then:

$$Spp_\psi(\mathbf{Q}_X^H[2], p \circ \phi, x) = -(0,0) + \sum_{i=1}^{n-1} \left(2 - \left\{\frac{-i}{n}\right\} - \left\{\frac{qi}{n}\right\}, 2 - \delta\left(\frac{1-q}{n} i\right)\right).$$

Remark 6.4. Above, we computed the spectral pairs of a composed function f. By our general result, we can compute $Spp_\psi(\mathbf{Q}_X^H[2], f, x)$ of an arbitrary function $f : (X, x) \to (\mathbf{C}, 0)$, provided that we know the decorated resolution graph of f (cf. the second part of this section).

6.2. The case of the trivial mixed Hodge module. Let (X,x) be a normal surface singularity and $f:(X,x)\to(\mathbf{C},0)$ an analytic germ. Assume that $\mathcal{M}=\mathbf{Q}^H_X[2]$. The limit $L\mathcal{M}$ is the one-dimensional mixed Hodge structure $\mathbf{Q}^H$ with trivial action and $Gr^W_l\mathbf{Q}^H=Gr^p_F\mathbf{Q}^H=0$ if $l\neq 0$ or $p\neq 0$. Denote the set of spectral pairs merely by $Spp_\psi(f)$. By Theorem 5.2 one has: $Spp_\psi(f)=Spp_\Gamma(\mathbf{Q}^H)$, where Γ is the decorated resolution graph of f.

Let $h=rank\,H_1(\Gamma)$ be the number of independent cycles of Γ. Alternatively, $h=rank\,H_1(E)-rank\,H_1(\tilde{E})$, where $\tilde{E}$ is a smooth model for $E=\pi^{-1}(x)$.

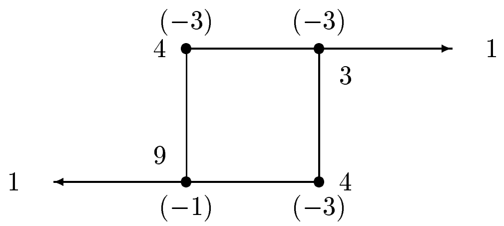
By a computation, we obtain an "almost symmetric" form of $Spp_\psi(f)$:

Proposition 6.5. *Let* $g=\sum_{d\in\mathcal{V}}g_d$, $R^k_d=\sum_{v\in\mathcal{V}_d}\{k\cdot m_v/m_d\}$, *and* $R^*(n)=\{1,\dots n-1\}$. *Then:*

$$\begin{aligned}
Spp_\psi(f) &= (h-1)(0,0)+(h+\#\mathcal{S}-1)(1,2)+g(0,1)+g(1,1)\\
&\quad+\sum_{s\in\mathcal{S}}\sum_{k\in R^*(m(e_s))}(1+\frac{k}{m(e_s)},2)+\sum_{e\in\tilde{\mathcal{E}}}\sum_{k\in R^*(m_e)}((\frac{k}{m_e},0)+(2-\frac{k}{m_e},2))\\
&\quad-\sum_{d\in\mathcal{D}}\sum_{\substack{k\in R^*(m_d)\\ R^k_d=0}}((\frac{k}{m_d},0)+(2-\frac{k}{m_d},2))\\
&\quad+\sum_{d\in\mathcal{D}}\sum_{\substack{k\in R^*(m_d)\\ R^k_d\neq 0}}(R^k_d-1)\,((\frac{k}{m_d},1)+(2-\frac{k}{m_d},1))\\
&\quad+\sum_{d\in\mathcal{D}}\sum_{k\in R^*(m_d)}g_d((\frac{k}{m_d},1)+(2-\frac{k}{m_d},1)).
\end{aligned}$$

Here $m(e_s)=g.c.d.(m(s),m(d_s))$, *where* d_s *is the unique vertex in* $\mathcal{D}$ *with* $(s,d_s)\in\mathcal{E}$.

Example 6.6. Let $(X,x)=\{z^2=(x+y^2)(x^2+y^7)\}\subset(\mathbf{C}^3,0)$ and $f(x,y,z)=z$. Then the decorated resolution graph is the following:



In parentheses are the numbers e_d, the others are the multiplicities. All exceptional divisors are rational. The spectral pairs are:

$$Spp_\psi(f)=\sum_{k=1}^{8}(2k/9,1)+(2/3,1)+(4/3,1)+2\cdot(1,2).$$

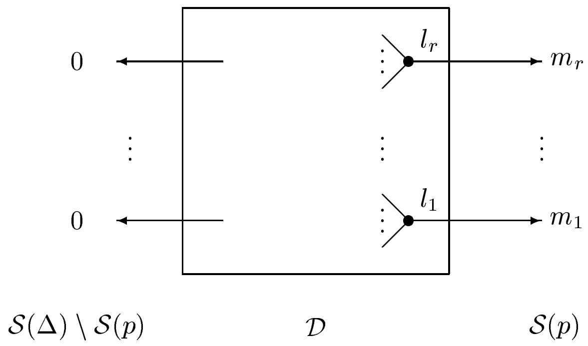
Remark 6.7. Using (3) one has: $Spp_\varphi(f)=Spp_\psi(f)+(0,0)$.

7. Topological Series of Curve Singularities

Let $p : (\mathbf{C}^2, 0) \to (\mathbf{C}, 0)$ be a curve singularity with irreducible decomposition $p = \prod_{j=1}^{r} p_j^{m_j}$, and $\Delta \subset (\mathbf{C}^2, 0)$ a reduced one-dimensional analytic space-germ with irreducible decomposition $\cup_{i=1}^{s} \Delta_i$. Let Γ be the decorated resolution graph of $p^{-1}(0) \cup \Delta$. Let $\mathcal{S}(\Delta)$ (resp. $\mathcal{S}(p)$) be the subset of $\mathcal{S}$ corresponding to the strict transforms $St(\Delta_i)$ (resp. $St(p_j^{-1}(0))$). Recall that the multiplicity of a vertex $v \in \mathcal{V}$ is the multiplicity of $p \circ \pi$ on the divisor D_v. In particular, the multiplicities of the vertices $v \in \mathcal{S}$ are: $\{m_j\}_{j=1}^{r}$ corresponding to $s \in \mathcal{S}(p)$, and $m_s = 0$ corresponding to $s \in \mathcal{S}(\Delta) \setminus \mathcal{S}(p)$ In the sequel we use the index notation $j \in \{1, \ldots, r\}$ for the set $\mathcal{S}(p)$.

The schematic graph of Γ, together with multiplicities, is:

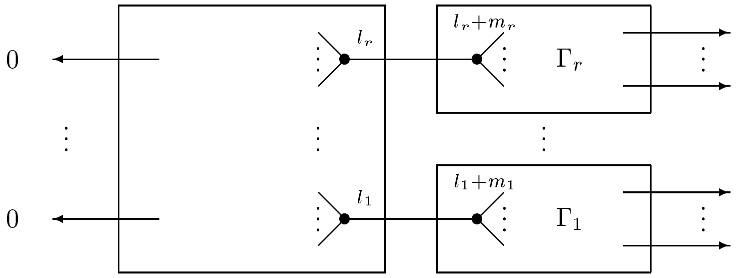
(12)



The vertices indexed by $\mathcal{D}$ are in the box. Those indexed by $\mathcal{S}$ are drawn as arrows. The arrows from the left-hand side corresponds to $\mathcal{S}(\Delta) \setminus \mathcal{S}(p)$, those from the right-hand side to $\mathcal{S}(p)$.

For each $j \in \mathcal{S}(p)$, let $d_j \in \mathcal{D}$ be the unique vertex which is adjacent to j. The corresponding exceptional divisor (D_{d_j}) is denoted by E_j and its multiplicity is $l_j = m_{d_j}$. The intersection point $D_j \cap E_j$ is denoted by P_j.

Definition 7.1. The topological series of curve singularities belonging to p, relative to Δ, consist of all curve singularities p' such that there are decorated resolution graphs Γ of $p^{-1}(0) \cup \Delta$, and Γ' of $(p')^{-1}(0) \cup \Delta$ respectively, such that Γ has form (12), and Γ' has the following form:



The index sets $\mathcal{D}', \mathcal{S}'$ and $\mathcal{E}'$ are defined similarly as $\mathcal{D}, \mathcal{S}$ and $\mathcal{E}$. For $j \in \{1, \dots, r\}$, the new index set of exceptional divisors (resp. of strict transforms or arrows) of Γ_j is denoted by $\mathcal{D}_j$ (resp. $\mathcal{S}_j$). Therefore $\mathcal{D}' = \mathcal{D} \sqcup (\sqcup_j \mathcal{D}_j)$, $\mathcal{S}' = (\mathcal{S} \setminus \mathcal{S}(p)) \sqcup (\sqcup_j \mathcal{S}_j)$, and $\mathcal{E}' = \mathcal{E} \sqcup (\sqcup_j \mathcal{E}_j) \sqcup (\sqcup_j \{e_j\})$, where $\mathcal{E}_j$ are the edges in Γ_j, and e_j is the new edge which joins Γ and Γ_j.

It can happen that the box Γ_j is empty. In that case, instead of Γ_j we have just an arrow as in the case of Γ.

7.1. Geometric meaning. Let U_j be a small neighbourhood of $P_j = D_j \cap E_j$. Fix a coordinate system (x, y) in U_j so that: $U_j \cap D_j = \{y = 0\}$, $U_j \cap E_j = \{x = 0\}$ and $p \circ \pi | U_j = x^{l_j} y^{m_j}$.

Definition 7.1 has the following meaning: $p' \circ \pi | U_j = \tilde{p}_j(x, y) x^{l_j}$, where $\tilde{p}_j$ is a curve singularity at P_j such that:

a) the line $\{x = 0\}$ is not in the tangent cone of $\{\tilde{p}_j = 0\}$,

b) the multiplicity of $\tilde{p}_j$ at P_j is m_j.

If $\tilde{p}_j = \prod_{k=1}^{r_j} p_{jk}^{m_{jk}}$ is the irreducible decomposition of $\tilde{p}_j$, then $\sum_k m_{jk} (x, p_{jk})_{P_j} = m_j$. Here $(\cdot, \cdot)_{P_j}$ denotes the intersection multiplicity at the point P_j.

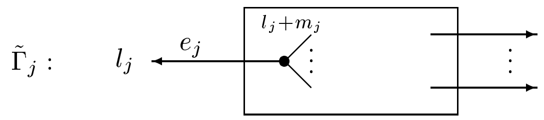
We define the numbers $\{\epsilon_j\}_{j=1}^r$ as follows. If $j \in \mathcal{S}(p) \setminus \mathcal{S}(\Delta)$, then the index set $\mathcal{S}_j$ is exactly $\{1, \dots, r_j\}$; we set $\epsilon_j = 0$. If $j \in \mathcal{S}(p) \cap \mathcal{S}(\Delta)$ (i.e., $\{p_j = 0\} = \Delta_{s_j}$), then one of the following conditions must hold:

1. $St(\Delta_{s_j}) = \{p_{jk} = 0\}$ for some k (i.e., $p_{jk}(x, y) = y$), or
2. $St(\Delta_{s_j}) \not\subset \{\tilde{p}_j = 0\}$.

In the first case, $\mathcal{S}_j = \{1, \dots, r_j\}$, and we set $\epsilon_j = 0$. In the second case, $\mathcal{S}_j = \{1, \dots, r_j\} \cup \{s_j\}$, and we take $\epsilon_j = 1$. This shows, that the index set $\mathcal{S}'(\Delta) \setminus \mathcal{S}'(p')$ (which is crucial in the main theorem 5.2) is $(\mathcal{S}(\Delta) \setminus \mathcal{S}(p)) \sqcup \sqcup_{j : \epsilon_j = 1} \{s_j\}$.

It is clear, that the diagram Γ_j is a part of the decorated resolution graph $\tilde{\Gamma}_j$ of $p' \circ \pi | U_j$ (relative to $\{xy = 0\}$). Actually, the resolution graph $\tilde{\Gamma}_j$ can be obtained from Γ_j by adding an arrow corresponding to the strict transform of $\{x = 0\} = E_j \cap U_j$. The arrow takes the place of the new edge e_j, which in Γ' joins Γ and Γ_j. Notice, that the multiplicity of (the strict transform of) E_j is l_j.

(13)



REMARK 7.2. A compatible definition of the topological series of curve singularities can be given using Eisenbud-Neumann diagrams [3] (see [14] in the case $\Delta = \emptyset$, and [6] in the general case).

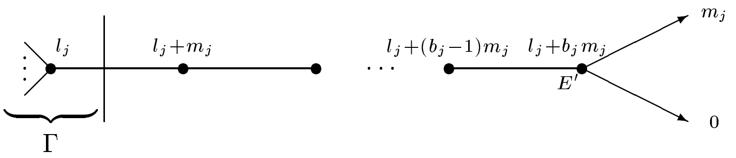
7.2. Topologically trivial series.

Definition 7.3. [6] We say that p' is an element of the "topologically trivial series" belonging to p, relative to Δ, if for any $j \in \{1, \dots, r\}$ either $\tilde{p}_j(x, y) = y^{m_j} a(x, y)$ or $\tilde{p}_j(x, y) = (y + x^{b_j} b(x, y))^{m_j} a(x, y)$ where $a, b \in \mathcal{O}_{U_j}^*$ and $b_j \geq 1$.

In the definition, "topologically trivial" means that the topological types of the germs p and p' are the same, but the positions of their zero set with respect to Δ are different.

In the first case ($\tilde{p}_j = y^{m_j}a$) one has $\epsilon_j = 0$ and the topological modification, even relative to Δ, is trivial. In the second case $\epsilon_j = 1$, and the diagram Γ_j is as follows:

(14)



7.3. Intrinsic invariants. Notice that the choice of the graphs Γ and Γ' is not unique. Therefore, the numbers b_j, l_j (or even the cycle $[M_{d_j}]$) have no intrinsic, geometric meaning. In our computations, we wish to express the spectral pairs in terms of some intrinsic invariants. Actually, this paragraph makes the connection between the set of numerical data used in the decoration of the resolution graph and those used in the Eisenbud-Neumann diagrams.

Fix $j \in \{1, \dots, r\}$ such that $\epsilon_j = 1$.

We define

$$n_j := \sum_{i \in \mathcal{S}(p)\setminus\{j\}} -A^{-1}(d_j, d_i)m_i,$$

where A is the intersection matrix of Γ as in Definition 4.1.

Let η_j be the multiplicity of p_j on E_j, i.e., $p_j \circ \pi | U_j = x^{\eta_j} y \cdot c(x,y),\ c \in \mathcal{O}^*_{U_j}$. Then the relation $[d_j] = \sum_{v\in\mathcal{S}} -A^{-1}(d_j, d_v)[v]$ gives a relation between the set of multiplicities of $p_j \circ \pi$ on the components of D, namely: $\eta_j = -A^{-1}(d_j, d_j)$. On the other hand, the same relation applied for the set of multiplicities of p gives: $l_j = n_j + \eta_j \cdot m_j$. Therefore:

$$b_j m_j + l_j = a_j m_j + n_j, \quad \text{where} \quad a_j := b_j + \eta_j. \tag{15}$$

Notice that $\mathcal{S}(p) = \mathcal{S}(p')$. Order the irreducible factors of p' in such a way that $p' = \prod_j (p'_j)^{m_j}$, and $Z(p'_j)$ in U_j is $\{y + x^{b_j} b(x,y) = 0\}$

Lemma 7.4. *The numbers a_j, n_j and $a_j m_j + n_j$ have the following expression in terms of intersection multiplicities at the origin:*

$$a_j = m(p_j, p'_j); \qquad n_j = m(p_j, \prod_{i\neq j} (p'_i)^{m_i});$$

$$a_j m_j + n_j = m(p_j, p').$$

Proof. $m(p_j, p'_j) = \deg(p_j | Z(p'_j)) = \deg(x^{\eta_j} y | y + x^{b_j} = 0) = b_j + \eta_j = a_j$. The others follow by similar argument. □

8. Topological Series of Plane Singularities with Coefficients in a Mixed Hodge Module

Let $\mathcal{M} \in MHM(\mathbf{C}^2, 0)$ be mixed Hodge module with singular locus Δ such that $\mathbf{V} = \mathcal{M}|_{B\setminus\Delta}$ is a polarisable variation of Hodge structure. Here (B, Δ) are some representatives as usual.

Consider a curve singularity p and let p' be an element of the topological series belonging to p, relative to Δ. Our purpose is to compare $Spp_\psi(\mathcal{M}, p, 0)$ and $Spp_\psi(\mathcal{M}, p', 0)$. Since p' is a modification of p in the neighbourhoods $\{U_j\}_{j=1}^r$, we expect that the difference of their spectral pairs depends only on the germs $\tilde{p}_j : (U_j, P_j) \to (\mathbf{C}, 0)$ and the restriction of $\mathcal{H}^0\pi^*\mathcal{M}$ on $\cup_{j=1}^r U_j$. Since the restriction on U_j has abelian representation over $U_j - \{xy = 0\}$, by our general principle, we expect that we need only the corresponding limit mixed Hodge structures at the points $\{P_j\}_j$ (and the graphs of the germs $\{\tilde{p}_j\}_j$).

8.1. Limit mixed Hodge structures. Let $\pi : U \to B$ be the resolution of $p^{-1}(0) \cup \Delta$ as in Section 7. Let $\pi^*\mathbf{V}$ be the lifting of $\mathbf{V}$ on $U \setminus D$ and $\mathbf{V}_j$ its restriction to $U_j \setminus D$ ($j \in \mathcal{S}(p)$). If $j \in \mathcal{S}(p) \setminus \mathcal{S}(\Delta)$, then $G_j := H_1(U_j \setminus D, \mathbf{Z}) = \mathbf{Z}$, and it is generated by $e_1 = [M_{d_j}]$. If $j \in \mathcal{S}(p) \cap \mathcal{S}(\Delta)$, then $G_j = \mathbf{Z}^2$, and it is generated by $e_1 = [M_{d_j}]$ and $e_2 = [M_j]$.

In both cases the variation $\mathbf{V}_j$ has abelian representation, therefore its limit mixed Hodge structure $L\mathbf{V}_j$ exists ([13]). Set $L_j\mathbf{V} := Gr^W L\mathbf{V}_j \in MHS(G_j)$. Its representation is denoted by ρ_j.

8.2. Intrinsic meaning. We show, that $L_j\mathbf{V}$ can be recovered in a more direct way, without using the resolution π.

Let $Z_j = \{p_j = 0\} \subset (\mathbf{C}^2, 0)$. Let $\sigma_j : (\mathbf{C}, 0) \to (Z_j, 0)$ be the normalization of Z_j, and let x be a local uniformization of $(\mathbf{C}, 0)$. Then $H_j := Gr^W \psi_x \mathcal{H}^0 \sigma_j^* \psi_{p_j} \mathcal{M} \in MHM(\{0\})$ is a mixed Hodge structure with two semi-simple monodromy actions: T_j^h (called the horizontal monodromy) is induced by ψ_{p_j} and T_j^v (called the vertical monodromy) is induced by ψ_x.

Now, if π is the resolution as above, then $\pi|_{St(Z_j)} : St(Z_j) \to Z_j$ is the normalization of Z_j. Moreover, if (x, y) are the coordinates in U_j as in Section 7.1, then x is a local coordinate in $St(Z_j)$. By our notation (cf. Section 7.3) $p_j \circ \pi | U_j = x^{\eta_j} y\, c(x, y)$, where $c \in \mathcal{O}^*_{U_j}$. Then by the properties of Section 2, $H_j = Gr^W \psi_x \psi_{p_j \circ \pi} \mathcal{N}$, where $\mathcal{N} = \mathcal{H}^0\pi^*\mathcal{M}$. Obviously, this depends only on $\mathcal{N}|_{U_j \setminus \{xy=0\}} = \mathbf{V}_j$ hence $H_j = Gr^W \psi_x \psi_{x^{\eta_j} yc} \mathbf{V}_j$.

Notice that $L\mathbf{V}_j = \psi_x \psi_y \mathbf{V}_j$, and this limit depends on the choice of the coordinates (x, y). The ambiguity is modulo the action of $\exp(\sigma_{\mathbf{C}})$, where $\sigma_{\mathbf{C}}$ is the complex monodromy cone. Therefore $L_j\mathbf{V} = H_j$.

By a local computation:

$$T_j^h = \rho_j([M_j]) \text{ and } T_j^v = \rho_j([M_{d_j}] - \eta_j[M_j]). \tag{16}$$

Notice, that if $j \in \mathcal{S}(p) \setminus \mathcal{S}(\Delta)$, then $T_j^h = \rho_j([M_j])$ is the identity and the only essential action is $T_j^v = \rho_j([M_{d_j}])$ (cf. Section 8.1).

8.3. Further computations. Similarly as in Section 8.2, we can consider the mixed Hodge structure $\tilde{H}_j = Gr^W \psi_x \mathcal{H}^0 \sigma_j^* \varphi_{p_j} \mathcal{M} \in MHM(\{0\})$ with semi-simple monodromies $\tilde{T}_j^h$ (induced by φ_{p_j}) and $\tilde{T}_j^v$ (induced by ψ_x).

Let $H_j = H_{j,\neq 1} \oplus H_{j,1}$ be the eigenspace decomposition given by T_j^h; similarly $\tilde{H}_j = \tilde{H}_{j,\neq 1} \oplus \tilde{H}_{j,1}$ provided by $\tilde{T}_j^h$.

On the other hand, if $\epsilon_j = 1$ (e.i. $Z_j = \Delta_{s_j}$, but $p'|_{\Delta_{s_j}} \not\equiv 0$), then $j \in \mathcal{S}(\Delta) \setminus \mathcal{S}(p')$; and to apply Theorem 5.2 we need the limit mixed Hodge structures $\mathbf{V}_j^k = Gr^W L\mathcal{H}^k i^!_{\Delta_{s_j}} \mathcal{M}$, too. Their monodromy is denoted by T_j^k.

We have the following relations between our limit mixed Hodge structures and their semi-simple actions:

Proposition 8.1. *Let $j \in \mathcal{S}(p) \cap \mathcal{S}(\Delta)$. Then:*

$$can : (H_{j,\neq 1}; T_j^h, T_j^v) \tilde{\to} (\tilde{H}_{j,\neq 1}; \tilde{T}_j^h, \tilde{T}_j^v) \quad \text{is an isomorphism in } MHS(\mathbf{Z}^2), \text{ and}$$

$$0 \to (V_j^0, T_j^0) \to (\tilde{H}_{j,1}, \tilde{T}_j^v) \xrightarrow{Var} (H_{j,1}, T_j^v)(-1) \to (V_j^1, T_j^1) \to 0$$

is an exact sequence in $MHS(\mathbf{Z})$.

Proof. Use (1), (2) and (3) and the fact that the vanishing cycle functor is exact. $\square$

9. The Spectral Pairs of Series of Plane Singularities

9.1. General formula. Let $\mathcal{M} \in MHM(\mathbf{C}^2, 0)$ be a polarizable mixed Hodge module with critical locus Δ. For simplicity, we assume that $\mathcal{M}$ restricted to the complement of Δ is pure (but this is not essential because of Lemma 2.9.c). Let $p : (\mathbf{C}^2, 0) \to (\mathbf{C}, 0)$ be a curve singularity, and p' a germ in the topological series belonging to p relative to Δ.

Fix $j \in \mathcal{S}(p)$. In Section 8.1 we constructed a representation $(H_j, \rho_j) \in MHS(G_j)$. If $\tilde{\Gamma}_j$ denotes the decorated resolution graph of $p' \circ \pi | U_j : (U_j, P_j) \to (\mathbf{C}, 0)$ (relative to $\{xy = 0\}$), and e_j its edge which joins Γ and Γ_j (cf. Section 7.1), then $Spp_{\tilde{\Gamma}_j}(\rho_j)$ and $Spp_{e_j}(\rho_j)$ are defined (cf. Definitions 4.3–4.4). (The latter one is $\sum_{p,q,i} T(p,q) Spp_{e_j}(\chi_i^{pq})$, with the notations of Definition 4.4.) We let

$$Spp_{\tilde{\Gamma}_j, e_j}(\rho_j) := Spp_{\tilde{\Gamma}_j}(\rho_j) + Spp_{e_j}(\rho_j).$$

Theorem 9.1. *Let ϵ_j and s_j be as in Section 7.1. Then one has:*

$$Spp_\psi(\mathcal{M}, p', 0) - Spp_\psi(\mathcal{M}, p, 0) = \sum_{j \in \mathcal{S}(p)} C_j, \ \text{where}$$

$$C_j = Spp_{\tilde{\Gamma}_j, e_j}(\rho_j) + \epsilon_j \sum_k (-1)^k c_{\deg(p'|\Delta_{s_j})} Spp(V_j^k, T_j^k).$$

Notice also that $\deg(p'|\Delta_{s_j})$ is equal to the intersection multiplicity $m_{P_j}(y, p' \circ \pi | U_j)$.

Proof. Using (2) and Lemma 2.9 we can assume that $\mathcal{M} = j_* \mathbf{V}$, where $j : B \setminus \Delta \to B$ is the inclusion. Consider the resolution π' corresponding to the graph Γ'. Notice that in the first step of Section 5.1 we did not use the commutativity assumption. Therefore, the relation (6) holds for both p and p'. If $d \in \mathcal{D}$ (or $e \in \tilde{\mathcal{E}}$) then in a neighbourhood of D_d (or of P_e, respectively) the functions $p' \circ \pi$ and $p \circ \pi$ differ only by an invertible function. Therefore, their contributions are equal. Using again the first step of Section 5.1 applied to $p' \circ \pi | U_j$, one has:

$$Spp(i_0^* \psi_{p'} \mathcal{M}) - Spp(i_0^* \psi_p \mathcal{M}) = Spp_{e_j}(\rho_j) + Spp(i_{P_j}^* \psi_{p' \circ \pi} \pi^* \mathcal{M}).$$

Now, since G_j is abelian, the result follows from Theorem 5.1. $\square$

We end this subsection with the following remark. By Lemma 2.9.e, one has:

$$Spp_\varphi(\mathcal{M}, p', 0) - Spp_\varphi(\mathcal{M}, p, 0) = Spp_\psi(\mathcal{M}, p', 0) - Spp_\psi(\mathcal{M}, p, 0)$$

for any $\mathcal{M}$, p and p'.

9.2. The case of topologically trivial series. In the sequel δ denotes the characteristic map of $\mathbf{R}\setminus\mathbf{Z}$, i.e., $\delta:\mathbf{R}\to\{0,1\}$ is defined by $\delta(x)=0$ if $x\in\mathbf{Z}$ and $=1$ otherwise.

Fix a character $\chi:\mathbf{Z}^2\to\mathbf{C}^*$ of finite order. Choose ω and λ such that $\chi(e_1)=\exp(2\pi i\omega)$ and $\chi(e_2)=\exp(2\pi i\lambda)$. Given a triple (n,m,a) of integers (where $n\geq 0$, $m>0$ and $a>0$), we define in $\mathbf{Z}[\mathbf{Q}\times\mathbf{Z}]$ the elements $C_\psi(\chi)$ and $C_\varphi(\chi)$ as follows.

For any $k\in\{0,\dots,n+am-1\}$ we let

$$\xi_k=\frac{k+\lambda+a\omega}{n+am}.$$

Then define $C_\psi(\chi):=$

$$\sum_{k=0}^{n+am-1}(1-\{-\omega\}+\{\xi_k\}+\{-\omega+m\xi_k\}-\{m\xi_k\},$$
$$2-\delta(\omega)-\delta(m\xi_k)+\delta(-\omega+m\xi_k)),$$

and define $C_\varphi(\chi):=$

$$\sum_{k=0}^{n+am-1}(\{\omega\}+\{\xi_k\}+\{-\omega+m\xi_k\}-\{m\xi_k\},\delta(\omega)-\delta(m\xi_k)+\delta(-\omega+m\xi_k)).$$

For $\rho\in MHS(\mathbf{Z}^2)$ define

$$C_\psi(\rho):=\sum_{p,q}\sum_{i=1}^{d(p,q)}T(p,q)C_\psi(\chi_i^{pq})$$

(and similarly $C_\varphi(\rho)$), where the characters χ_i^{pq}'s are as in Definition 4.4.

Example 9.2. 1) Assume that $m=1$. Since ω and λ are defined modulo an integer, we can take some representatives $\omega\in[0,1)$ and $\lambda\in[n\omega,n\omega+1)$. Then

$$C_\varphi(\chi)=\sum_{k=0}^{n+a-1}(\xi_k,\delta(\omega)-\delta(\xi_k)+\delta(-\omega+\xi_k)).$$

2) Assume that $m=1$ and $n=0$. Then

$$C_\varphi(\chi)=\sum_{k=0}^{a-1}\left(\omega+\frac{k+\lambda}{a},\delta(\omega)-\delta\left(\omega+\frac{k+\lambda}{a}\right)+\delta\left(\frac{k+\lambda}{a}\right)\right).$$

Now we return to our representations (H_j,ρ_j) and $(\tilde{H}_j,\tilde{\rho}_j)$, $(j\in\mathcal{S}(p))$. Recall that in $G_j=\mathbf{Z}^2$ one has: $e_1=[M_{d_j}]$ and $e_2=[M_j]$. We define the representation $\rho_j^*:\mathbf{Z}^2\to Aut_{MHS}(H_j)$ by the following change of bases: $\rho_j^*(e_1)=\rho_j(e_2)$ and $\rho_j^*(e_2)=\rho_j(e_1-\eta_j e_2)$. Then $\rho_j^*(e_1)=T_j^h$ and $\rho_j^*(e_2)=T_j^v$ by (16). Similarly we define $\tilde{\rho}_j^*$.

The relation between the above invariants is given in the following lemma.

Lemma 9.3. *Let (H_j,ρ_j), $(\tilde{H}_j,\tilde{\rho}_j)\in MHS(\mathbf{Z}^2)$ and (V_j^k,T_j^k) be as in Section 8. Define C_ψ and C_φ using the triplet (n_j,m_j,a_j). Then*

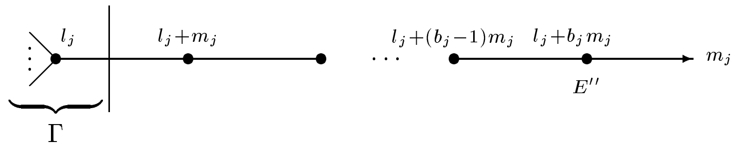
$$C_\psi(\rho_j^*)+\sum_k(-1)^k c_{n_j+a_jm_j}Spp(V_j^k,T_j^k)=C_\varphi(\tilde{\rho}_j^*).$$

Proof. Notice that if $\chi(e_1) \neq 1$ (i.e., $\omega \notin \mathbf{Z}$) then $C_\psi(\chi) = C_\varphi(\chi)$. Otherwise, $C_\varphi(\chi) = T(-1,-1)C_\psi(\chi) = c_{n+am}Spp(\mathbf{C}, \chi(e_2))$. Now use Proposition 8.1. $\square$

Now we consider p', a germ in the topological series belonging to p relative to Δ, as in Section 7.2. From the first step of Section 5.1, it is clear that the correction C_j is zero provided that $j \in \mathcal{S}(p) \setminus \mathcal{S}(\Delta)$. In the sequel we assume that $j \in \mathcal{S}(p) \cap \mathcal{S}(\Delta)$. We also assume that $\tilde{p}_j = (y + x^{b_j}b(x,y))a(x,y)$; otherwise $C_j = 0$ again.

We will compute C_j in a more direct way. First notice that our relations do not depend on the choice of the resolution graph. In the case of p', we will take the graph Γ', but for p we will use the graph of π modified by b_j blowing-ups. This graph will be denoted by Γ'', and its schematic form is the following:

(17)



The arrow corresponds to the strict transform of $Z_j = \Delta_{s_j}$.

By a similar argument to that in the proof of Theorem 9.1, one has:

$$C_j = Spp_{E'}(\rho_j) - Spp_E(\rho_j) + \sum_k (-1)^k c_{n_j+a_jm_j} Spp(V_j^k, T_j^k), \tag{18}$$

where the vertices E'' and E' are drawn on the corresponding graphs, (cf. (17) and (14)).

Lemma 9.4. $Spp_{E'}(\rho_j) - Spp_E(\rho_j) = C_\psi(\rho_j^*)$.

Proof. We can assume that ρ_j is one dimensional. Let $\rho_j^*(e_1) = \exp(2\pi i\omega)$ and $\rho_j^*(e_2) = \exp(2\pi i\lambda)$.

In the case of p (graph Γ'') the invariants are:

- the degree of E'' is 2;
- the multiplicities of the adjacent vertices are: $n_j + (a_j-1)m_j$ and m_j;
- the monodromies around the adjacent vertices are: $\exp(2\pi i(\lambda + (a_j-1)\omega))$ and $\exp(2\pi i\omega)$.

In the case of p' (graph Γ') the invariants are:

- the degree of E' is 3;
- the multiplicities of the adjacent vertices are: $n_j + (a_j-1)m_j$, m_j and 0;
- the monodromies around the adjacent vertices are: $\exp(2\pi i(\lambda + (a_j-1)\omega))$, 1 and $\exp(2\pi i\omega)$.

The multiplicity of E' and E'' is equal to $n_j + a_jm_j$, and the monodromy around them is $\exp(2\pi i(\lambda + a_j\omega))$.

The verification is left to the reader. $\square$

The main result of this section is the following:

Theorem 9.5. *Let $\mathcal{M}$ be a pure mixed Hodge module with critical locus Δ, p' a germ of the topologically trivial series belonging to $p = \prod_j p_j^{m_j}$, relative to Δ. Let*

$\tilde{H}_j$ be the limit mixed Hodge structure of $\varphi_{p_j}\mathcal{M}$ with semisimple monodromies T_j^h (induced by φ_{p_j}) and T_j^v (induced by the monodromy of $\varphi_{p_j}\mathcal{M}$). Then

$$Spp_\psi(\mathcal{M},p',0) - Spp_\psi(\mathcal{M},p,0) = \sum_{j:\epsilon_j=1} C_{\varphi,j}(\tilde{H}_j;T_j^h,T_j^v),$$

where $C_{\varphi,j}$ is C_φ with parameters (n_j,m_j,a_j).

Proof. Use Theorem 9.1, (18), Lemma 9.4 and Lemma 9.3. □

Corollary 9.6. *Let $\mathcal{M}$ and p, p' as in Theorem 9.5. Assume that p is irreducible. Set*

$$(\tilde{H}_j;T_j^h,T_j^v) = \oplus_{p,q}\oplus_{i=1}^{d(p,q)}(H_i^{pq};\exp(2\pi i\omega_i^{pq}),\exp(2\pi i\lambda_i^{pq})),$$

where H_i^{pq} is one-dimensional $\mathbf{C}$-Hodge structure of type (p,q), and $\omega_i^{pq},\lambda_i^{pq}\in[0,1)$. The integer a denotes the intersection multiplicity $m(p,p')$. Then:

$$Spp_\psi(\mathcal{M},p',0) - Spp_\psi(\mathcal{M},p,0) =$$
$$\sum_{p,q,i} T(p,q)\sum_{k=0}^{a-1}\left(\omega_i^{pq}+\frac{k+\lambda_i^{pq}}{a},\delta(\omega_i^{pq})-\delta\left(\omega_i^{pq}+\frac{k+\lambda_i^{pq}}{a}\right)+\delta\left(\frac{k+\lambda_i^{pq}}{a}\right)\right).$$

Proof. If p is irreducible, then $\mathcal{S}(p)$ has only one element whose multiplicity is one, and $n=0$. Now apply Example 9.2. □

10. Spectral Pairs of Series of Composed Singularities

10.1. Some invariants of an ICIS with 2-dimensional base space. Let $\phi:(X,x)\to(\mathbf{C}^2,0)$ be an analytic germ such that both (X,x) and $(\phi^{-1}(0),0)$ are isolated complete intersection singularities, and $\dim X = n\geq 2$. By [18] there exists a projective extension $\bar{\phi}:\bar{X}\to(\mathbf{C}^2,0)$ of ϕ such that the central fiber $\bar{X}_0=\bar{\phi}^{-1}(0)$ has only x as singular point and the germ $\bar{\phi}:(\bar{X},x)\to(\mathbf{C}^2,0)$ is analytically equivalent to ϕ. Fix a good representative $\bar{\phi}:\bar{X}\to B$ of $\bar{\phi}$ such that a suitable restriction of it is a good representative $\phi:X\to B$ of ϕ. We let $C\subset X$ be the critical locus of ϕ and $\Delta=\phi(C)$ the (reduced) discriminant locus with irreducible decomposition $\cup_{i=1}^s\Delta_i$. Over Δ_i lie the irreducible components $C_{i1},\dots,C_{it_i}$ of C. The degree of the rectriction $\phi:C_{il}\to\Delta_i$ is d_{il}.

Let $\sigma_i:U\to\Delta_i$ (resp. $\sigma_{il}:U\to C_{il}$) be the normalization and z_i (resp. z_{il}) a uniformizing parameter in U.

Let p_i be a generator of the ideal of Δ_i. Then the support of the mixed Hodge module $\varphi_{p_i\circ\bar{\phi}}\mathbf{Q}_{\bar{X}}^H[n]$ is a subset of $\bar{X}_0\cup C$. For any $i\in\{1,\dots,s\}$ and $l\in\{1,\dots,t_i\}$ we define $(\tilde{H}_{il};\tilde{T}_{il}^h,\tilde{T}_{il}^v)\in MHS(\mathbf{Z}^2)$ by

$$\tilde{H}_{il} := Gr^W\psi_{z_{il}}H^0\sigma_{il}^*\varphi_{p_i\circ\bar{\phi}}\mathbf{Q}_{\bar{X}}^H[n].$$

The (horizontal) monodromy T_{il}^h is induced by $\varphi_{p_i\circ\bar{\phi}}$, the other (vertical) one T_{il}^v by $\psi_{z_{il}}$. By the properties of the vanishing cycle functors, they commute.

The structure $(\tilde{H}_{il},\tilde{T}_{il}^h)$ has another interpretation too. For this, take a point $P\in\Delta_i-\{0\}$ and a transversal slice S to Δ_i at P. Choose $P'\in\phi^{-1}(P)\cap C_{il}$. Then $\phi:(\phi^{-1}(S),P')\to(S,P)$ defines an isolated hypersurface singularity. Its limit mixed Hodge structure on its reduced cohomology is exactly $\tilde{H}_{il}$ and $\tilde{T}_{il}^h$ is the semi-simple part of its monodromy.

If H is a mixed Hodge structure with commuting automorphisms T^h and T^v, define $c_m(H)$ as $\oplus_{i=1}^m H$ with automorphisms:

$$c_m(T^h)(x_1,\dots,x_m) = (T^h(x_1),\dots,T^h(x_m)), \text{ and}$$
$$c_m(T^v)(x_1,\dots,x_m) = (T^v(x_m),x_1,\dots,x_{m-1}).$$

The new structure in $HMS(\mathbf{Z}^2)$ is denoted by $c_m(H;T^h,T^v)$.

Define the mixed Hodge module $\mathcal{M} \in MHM(\mathbf{C}^2,0)$ by $\mathcal{M} = \mathcal{H}^0\bar{\phi}_*\mathbf{Q}^H_{\bar{X}}[n]$. Then the critical locus of $\mathcal{M}$ is Δ and $\mathcal{M}|_{B\setminus\Delta}$ is pure. In Section 8.3 we associated the set of invariants $(\tilde{H}_i,\tilde{T}^h_i,\tilde{T}^v_i)_{i=1}^s \in MHS(\mathbf{Z}^2)$ with such a module, namely $\tilde{H}_i = Gr^W\psi_{z_i}\mathcal{H}^0\sigma_i^*\varphi_{p_i}\mathcal{M}$. The horizontal monodromy $\tilde{T}^h_i$ is induced by φ_{p_i}, the vertical one $\tilde{T}^v_i$ by ψ_{z_i}.

Lemma 10.1. *The following isomorphism holds:*

$$(\tilde{H}_i;\tilde{T}^h_i,\tilde{T}^v_i) = \oplus_{l=1}^{t_i} c_{d_{il}}(\tilde{H}_{il};\tilde{T}^h_{il},\tilde{T}^v_{il}).$$

Proof. Use (4). □

10.2. Topological series of composed singularities. Let ϕ be as above and $p:(\mathbf{C}^2,0)\to(\mathbf{C},0)$ an arbitrary analytic germ.

Definition 10.2. The topological series of composed singularities belonging to $f = p\circ\phi$ consists of all composed singularities $f' = p'\circ\phi$ such that p' is in the topological series of curve singularities belonging to p relative to the discriminant locus Δ of ϕ.

For simplicity, we will consider only germs $f' = p'\circ\phi$, where p' belongs to a topologically trivial series of p, relative to Δ. For these germs we extend Theorem 9.5 and Corollary 9.6. The interested reader can formulate easily the corresponding result which extends Theorem 9.1 and holds for any p'.

We recall our notations. Let $p = \prod_{j=1}^r p_j^{m_j}$ be the irreducible decomposition of p. Let p' be as above. We write $p' = \prod_{j=1}^r (p'_j)^{m_j}$, where we ordered the irreducible factors of p' as in Lemma 7.4. For any $j\in\mathcal{S}(p)\cap\mathcal{S}(\Delta)$ (with $\epsilon_j = 1$) consider the numerical invariants

$$a_j = m(p_j,p'_j) \text{ and } n_j = m(p_j,\prod_{i\neq j}(p'_i)^{m_i}).$$

Theorem 10.3. *With the notation $Spp_\psi(f) = Spp_\psi(\mathbf{Q}^H_X[n],f,x)$ one has:*

$$Spp_\psi(f') - Spp_\psi(f) = \sum_{j\in\mathcal{S}(p)\cap\mathcal{S}(\Delta)} \epsilon_j\; C_{\varphi,j}(\tilde{H}_j;\tilde{T}^h_j,\tilde{T}^v_j),$$

where $C_{\varphi,j}$ is C_φ with parameters (n_j,m_j,a_j).

Remark 10.4. Lemma 10.1 implies that

$$C_{\varphi,j}(\tilde{H}_j:\tilde{T}^h_j,\tilde{T}^v_j) = \sum_{l=1}^{t_i} C_\varphi(n_jd_{jl},m_j,a_jd_{jl})(\tilde{H}_{jl};\tilde{T}^h_{jl},\tilde{T}^v_{jl}).$$

Proof of Theorem 10.3. First notice, that $Spp_\psi(f') - Spp_\psi(f) = Spp_\varphi(f') - Spp_\varphi(f)$ (by Lemma 2.9.e).

Define $\mathcal{N} = i^*_{\bar{X}_0}\varphi_{p\circ\bar{\phi}}\mathbf{Q}^H_{\bar{X}}[n]$. Let $j:\bar{X}_0\setminus\{x\}\hookrightarrow\bar{X}_0$ be the natural inclusion. Then $j^*\mathcal{N} = j^*\varphi_{p\circ\bar{\phi}}\mathbf{Q}^H_{\bar{X}}[n]$, because $\varphi_{p\circ\bar{\phi}}\mathbf{Q}^H_{\bar{X}}[n]$ is supported in $\bar{X}_0\cup C$; and $Gr^W j^*\mathcal{N}$ is a constant variation of mixed Hodge structure with stalk isomorphic to $\varphi_p \mathbf{Q}^H_B[2]$. Similarly we consider $\mathcal{N}'$ defined by p'. Then $Gr^W j^*\mathcal{N} \simeq Gr^W j^*\mathcal{N}'$. Therefore, by the first distinguished triangle of (2), one has:

$$Spp_\varphi(f') - Spp_\varphi(f) = Spp(\bar{\phi}_*\mathcal{N}') - Spp(\bar{\phi}_*\mathcal{N}).$$

But $\bar{\phi}_*\mathcal{N} = i_0^*\varphi_p\bar{\phi}_*\mathbf{Q}^H_X$. Since for $j \neq 0$ the module $\mathcal{H}^j\bar{\phi}_*\mathbf{Q}^H_X$ is smooth

$$Spp\varphi_p\mathcal{H}^j\bar{\phi}_*\mathbf{Q}^H_X = Spp\varphi_{p'}\mathcal{H}^j\bar{\phi}_*\mathbf{Q}^H_X$$

by Theorem 5.1. Therefore

$$Spp_\varphi(f') - Spp_\varphi(f) = Spp(\mathcal{M}, p', 0) - Spp(\mathcal{M}, p, 0).$$

Now apply Theorem 9.5. □

Corollary 10.5. *Assume that p is irreducible, and $\{p = 0\} = \Delta_j$. Let*

$$(\tilde{H}_{jl}; \tilde{T}^h_{jl}, \tilde{T}^v_{jl}) = \oplus_{p,q} \oplus_{i=1}^{d(p,q,l)} (H_i^{pq,l}; \exp(2\pi i\omega_i^{pq,l}), \exp(2\pi i\lambda_i^{pq,l}))$$

for any $l = 1, \dots, t_j$, where $H_i^{pq,l}$ is one-dimensional $\mathbf{C}$-Hodge structure of type (p, q), and $\omega_i^{pq,l}, \lambda_i^{pq,l} \in [0, 1)$. Let d_l be the degree of $\phi : C_{jl} \to \Delta_j$ and $a = m(p, p')$. Then $Spp_\psi(f') - Spp_\psi(f) =$

$$\sum_{l,i,p,q} T(p,q) \sum_{k=0}^{ad_l - 1} (\omega_i^{pq,l} + \frac{k + \lambda_i^{pq,l}}{ad_l}, \delta(\omega_i^{pq,l}) - \delta(\omega_i^{pq,l} + \frac{k + \lambda_i^{pq,l}}{ad_l}) + \delta(\frac{k + \lambda_i^{pq,l}}{ad_l})).$$

Proof. Apply Corollary 9.6 and Remark 10.4. □

Example 10.6. The Yomdin's series (see [21, 15, 19, 11]). Let $f : (\mathbf{C}^n, 0) \to (\mathbf{C}, 0)$ $(n \geq 2)$ be an analytic germ with one-dimensional critical locus Σ, which has an irreducible decomposition $\cup_{l=1}^t \Sigma_l$. The mixed Hodge module $\varphi_f \mathbf{Q}^H_X[n]$ is supported on Σ and its restriction on $\Sigma_l \setminus \{0\}$ is an admissible variation of mixed Hodge structure. Denote its limit by $\tilde{K}_l$. Two natural semi-simple automorphisms act on $\tilde{K}_l$: $\tilde{T}^h_l$ induced by φ_f, and $\tilde{T}^v_l$ induced by the monodromy of the restriction $\varphi_f \mathbf{Q}^H_X[n] | \Sigma_l \setminus 0$.

Let $l : (\mathbf{C}^n, 0) \to (\mathbf{C}, 0)$ be a generic linear form. Then the pair $\phi := (f, l) : (\mathbf{C}^n, 0) \to (\mathbf{C}^2, 0)$ defines an ICIS, and $\phi(\Sigma) = \Delta_1$ is one of the irreducible components of Δ. Then

$$(\tilde{K}_l, \tilde{T}^h_l, \tilde{T}^v_l) = (\tilde{H}_{1l}, \tilde{T}^h_{1l}, \tilde{T}^v_{1l}).$$

Notice that $f = p \circ \phi$ where $p(z_1, z_2) = z_1$. Then for sufficiently large a $p'(z_1, z_2) = z_1 + z_2^a$ is in the topologically trivial series belonging to p, relative to Δ.

Therefore $Spp_\psi(f + l^a) - Spp_\psi(f)$ follows from Remark 10.4.

11. A Generalized Sebastiani-Thom Type Result

If $(H_i, W^i_\bullet, F^\bullet_i)$ $(i = 1, 2)$ are mixed Hodge structures, then $H_1 \otimes H_2$ carries a mixed Hodge structure with weight filtration $W_k = \sum_{p+q=k} W^1_p \otimes W^2_q$ and Hodge filtration $F^k = \sum_{p+q=k} F^p_1 \otimes F^q_2$. This mixed Hodge structure is still denoted by $H_1 \otimes H_2$.

A graded mixed Hodge structure is a finite direct sum $H^\bullet = \oplus_k H^k$ of mixed Hodge structures $\{H^k\}_k$. Their category is denoted by $GMHS$. There is a natural extension of the tensor product to $GMHS$ by $H^\bullet_1 \otimes H^\bullet_2 := \oplus_k(\oplus_{p+q=k} H^p_1 \otimes H^q_2)$.

There is a natural tensor product $MHS(G_1)\otimes MHS(G_2)\to MHS(G_1\times G_2)$, which associates with two elements, say $\rho_i: G_i\to Aut_{MHS}(H_i)$ $(i=1,2)$, an element $\rho=\rho_1\otimes\rho_2: G_1\times G_2\to Aut_{MHS}(H_1\otimes H_2)$ defined by $\rho(g_1,g_2)=\rho_1(g_1)\otimes\rho_2(g_2)$.

If $GMHS(G)$ denotes the graded version of $MHS(G)$ (i.e., the category of representations $\rho^\bullet=\oplus_k\rho^k$, where $\rho^k\in MHS(G)$), then the above tensor product has an extension

$$GMHS(G_1)\otimes GMHS(G_2)\to GMHS(G_1\times G_2)$$

given by $\rho_1^\bullet\otimes\rho_2^\bullet=\oplus_k(\oplus_{p+q=k}\rho_1^p\otimes\rho_2^q)$.

There is an extension $Spp: GMHS(\mathbf{Z})\to\mathbf{Z}[\mathbf{Q}\times\mathbf{Z}]$ of the map defined in Definition 2.6 by $Spp(\rho^\bullet)=\sum_k(-1)^k Spp(\rho^k)$. Moreover if Γ is as in Section 3, then Spp_Γ also extends by the same formula as above.

The map $(\alpha,w)*(\beta,\omega)=(\alpha+\beta,w+\omega)$ extends to a bilinear map $\mathbf{Z}[\mathbf{Q}\times\mathbf{Z}]\otimes_{\mathbf{Z}}\mathbf{Z}[\mathbf{Q}\times\mathbf{Z}]\to\mathbf{Z}[\mathbf{Q}\times\mathbf{Z}]$. In general it is not true that $Spp(\rho_1\otimes\rho_2)=Spp(\rho_1)*Spp(\rho_2)$ but still one has the following result.

Lemma 11.1. *Set $(H_i,\rho_i)\in MHS(\mathbf{Z})$ such that $\rho_1=1_{H_1}$. Let c_n $(n\in\mathbf{N}^*)$ be the map defined in Lemma 2.9.f. Then one has*:

a) $Spp(\rho_1\otimes\rho_2)=Spp(\rho_1)*Spp(\rho_2)$.

b) *If $\xi,\zeta\in\mathbf{Z}[\mathbf{Q}\times\mathbf{Z}]$ such that $\xi=\sum(\alpha,\omega)$ satisfies $\alpha\in\mathbf{Z}$, then*

$$c_n(\xi*\zeta)=\xi*c_n(\zeta).$$

The easy proof is left to the reader.

In the sequel it is convenient to define the map c_∞ as the zero map.

Let $g:(X,x)\to(\mathbf{C},0)$ be an analytic germ. Denote $\oplus_j\mathcal{H}^j i_0^*\psi_g\mathbf{Q}_X^H[n]$ by $(H_g^\bullet,T_g^\bullet)\in GMHS(\mathbf{Z})$. Here $T_g^\bullet=\rho_g^\bullet(1)$ is the semisimple part of the monodromy. The decomposition $\psi_g=\psi_{g,1}\oplus\psi_{g,\neq1}$ gives a decomposition $H_{g,1}^\bullet\oplus H_{g,\neq1}^\bullet$ of $H_g^\bullet$. Therefore the spectrum $Spp_\psi(g):=Spp(H_g^\bullet,T_g^\bullet)$ can be written as the sum of the corresponding spectral pairs $Spp_{\psi,1}(g)$ and $Spp_{\psi,\neq1}(g)$. There are similar notations for φ instead of ψ. By Lemma 2.9.e one has:

$$Spp_\psi(g)=Spp_\varphi(g)+(-1)^{n-1}(0,0).$$

We introduce the notation:

$$Spp_!(g)=Spp_{\varphi,1}(g)-T(1,1)Spp_{\psi,1}(g)=(1-T(1,1))Spp_{\psi,1}+(-1)^n(0,0).$$

Consider an analytic germ $p:(\mathbf{C}^2,0)\to(\mathbf{C},0)$ and the space-germ $(\Delta,0)$ given by $\{cd=0\}$, where (c,d) are the local coordinates in $(\mathbf{C}^2,0)$. Let Γ be the decorated resolution graph of p with respect to $(\Delta,0)$ (cf. Example 4.5). Notice that $G(\Gamma)=\mathbf{Z}^{\mathcal{S}}=H_1(\mathbf{C}^2\setminus((p^{-1}(0)\cup\Delta)),\mathbf{Z})$. The group $H_1(\mathbf{C}^2\setminus\Delta,\mathbf{Z})=\mathbf{Z}^2$ is generated by $[M_c]$ and $[M_d]$, where M_c (resp. M_d) is an oriented circle in a transversal slice to $\{c=0\}$ (resp. $\{d=0\}$).

Let $g:(X,x)\to(\mathbf{C},0)$ and $h:(Y,y)\to(\mathbf{C},0)$ be two analytic germs which define ICIS's, and $p:(\mathbf{C}^2,0)\to(\mathbf{C},0)$ an arbitrary analytic germ. Define $f:(X\times Y,x\times y)\to(\mathbf{C},0)$ by $f(x,y)=p(g(x),h(y))$.

Consider the tensor product $H_g^\bullet\otimes H_h^\bullet\in GMHS(\mathbf{Z}^2)$. By the identification of $\mathbf{Z}^2$ with $H_1(\mathbf{C}^2\setminus\Delta,\mathbf{Z})$ one has:

$$\rho^\bullet([M_c])=T_g^\bullet\otimes id \text{ and } \rho^\bullet([M_d])=id\otimes T_h^\bullet.$$

Via the map $G(\Gamma) \to H_1(\mathbf{C}^2 \setminus \Delta, \mathbf{Z})$, induced by the inclusion, $H_g^\bullet \otimes H_h^\bullet$ becomes an element of $GMHS(G(\Gamma))$.

Let $r(c)$ be the intersection multiplicity $m_0(p,c)$ if $\{c=0\}$ is not a factor of p, and $=\infty$ otherwise. Symmetrically define $r(d)$. The main result of this section is the following.

Theorem 11.2. *Let h, g, p and f be as above. Then:*

$$Spp_\psi(f) = Spp_\Gamma(H_h^\bullet \otimes H_g^\bullet) + c_{r(d)} Spp_\psi(g) * Spp_!(h) + c_{r(c)} Spp_\psi(h) * Spp_!(g).$$

Proof. Consider a projective extension $\bar{g} : (\bar{X}, \bar{X}_0) \to (\mathbf{C}, 0)$ of the germ g such that $x \in \bar{X}_0 = \bar{g}^{-1}(0)$ is the only singular point of $\bar{g}$ and $\bar{g} : (\bar{X}, x) \to (\mathbf{C}, 0)$ analytically equivalent to g. (For the existence of $\bar{g}$, see [18].) Similarly, let $\bar{h} : (\bar{Y}, \bar{Y}_0) \to (\mathbf{C}, 0)$ be a projective extension of h with the above properties. Now use several times the first distinguished triangle (2) applied to the natural stratification of $\bar{X}_0 \times \bar{Y}_0$, and apply Theorem 5.2. The details are left to the reader. $\square$

For the corresponding formula at the zeta function level, see [7, 8].

Example 11.3. Let g and h be as above. Then one has:

a) If $Spp_\psi(g) = \sum(\alpha, w)$ and $Spp(h) = \sum(\beta, \omega)$, then

$$Spp_\psi(g(x) \cdot h(y)) = [T(1,1) - 1] \cdot \sum_{[\alpha]=[\beta]} ([\alpha] + [\beta] + \alpha, w + \omega).$$

b) For the sum $f(x,y) = g(x) + h(y)$, it is simpler to rewrite Theorem 11.2 in terms of Spp_{St} (see Remark 2.7):

$$Spp_{St}(f(x) + g(y)) = T(1,0) Spp_{St}(g) * Spp_{St}(h)$$

(cf. [12]).

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