Conjugacy criteria for half-linear ODE in theory of PDE with generalized p-Laplacian and mixed powers

Robert Mařík

Dpt. of Mathematics
Mendel University
Brno, CZ

$$\operatorname{div}\left(A(x)\,\|\nabla y\|^{p-2}\,\nabla y\right) + \left\langle \vec{b}(x), \|\nabla y\|^{p-2}\,\nabla y\right\rangle + c(x)|y|^{p-2}y + \sum_{i=1}^{m} c_i(x)|y|^{p_i-2}y = e(x), \tag{E}$$

- $x = (x_1, \dots, x_n)_{i=1}^{n} \in \mathbb{R}^n$, $p > 1$, $p_i > 1$,
- $A(x)$ is elliptic $n \times n$ matrix with differentiable components, $c(x)$ and $c_i(x)$ are Hölder continuous functions, $\vec{b}(x) = \big(b_1(x), \dots, b_n(x)\big)$ is continuous n-vector function,
- $\nabla = \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right)_{i=1}^{n}$ and $\operatorname{div} = \frac{\partial}{\partial x_1} + \dots + \frac{\partial}{\partial x_n}$ is are the usual nabla and divergence operators,
- q is a conjugate number to the number p, i.e., $q = \frac{p}{p-1}$,
- $\langle \cdot, \cdot \rangle$ is the usual scalar product in $\mathbb{R}^n$, $\|\cdot\|$ is the usual norm in $\mathbb{R}^n$, $\|A\| = \sup\{\|Ax\| : x \in \mathbb{R}^n \text{ with } \|x\| = 1\} = \lambda_{\mathsf{max}}$ is the spectral norm
- **solution** of (E) in $\Omega \subseteq \mathbb{R}^n$ is a differentiable function $u(x)$ such that $A(x)\|\nabla u(x)\|^{p-2}\nabla u(x)$ is also differentiable and u satisfies (E) in Ω
- $S(a) = \{x \in \mathbb{R}^n : \|x\| = a\}$,
  $\Omega(a) = \{x \in \mathbb{R}^n : a \leq \|x\|\}$,
  $\Omega(a, b) = \{x \in \mathbb{R}^n : a \leq \|x\| \leq b\}$

$$u'' + c(x)u = 0 \tag{1}$$

- Equation (1) is oscillatory if each solution has infinitely many zeros in $[x_0, \infty)$.
- Equation (1) is oscillatory if each solution has a zero $[a, \infty)$ for each a.
- Equation (1) is oscillatory if each solution has conjugate points on the interval $[a, \infty)$ for each a.
- All definition are equivalent (no accumulation of zeros and Sturm separation theorem).
- Equation is oscillatory if $c(x)$ is large enough. Many oscillation criteria are expressed in terms of the integral $\int^{\infty} c(x)\,\mathrm{d}x$ (Hille and Nehari type)
- There are oscillation criteria which can detect oscillation even if $\int^{\infty} c(x)\,\mathrm{d}x$ is extremly small. These criteria are in fact series of conjugacy criteria.

$$(p(t)u')' + c(t)u + \sum_{i=1}^{m} c_i(t)|u|^{\alpha_i} \operatorname{sgn} u = e(t) \tag{2}$$

where $\alpha_1 > \cdots > \alpha_m > 1 > \alpha_{m+1} > \cdots > \alpha_n > 0$.

Theorem A (Sun,Wong (2007)). *If for any $T \geq 0$ there exists a_1, b_1, a_2, b_2 such that $T \leq a_1 < b_1 \leq a_2 < b_2$ and*

$$\begin{cases} c_i(t) \geq 0 & t \in [a_1, b_1] \cup [a_2, b_2], \; i = 1, 2, \ldots, n \\ e(x) \leq 0 & t \in [a_1, b_1] \\ e(x) \geq 0 & t \in [a_2, b_2] \end{cases}$$

and there exists a continuously differentiable function $u(t)$ satisfying $u(a_i) = u(b_i) = 0$, $u(t) \neq 0$ on (a_i, b_i) and

$$\int_{a_i}^{b_i} \left\{ p(t)u'^2(t) - Q(t)u^2(t) \right\} \mathrm{d}t \leq 0 \tag{3}$$

for $i = 1, 2$, where

$$Q(t) = k_0 |e(t)|^{\eta_0} \prod_{i=1}^{m} \Big(c_i^{\eta_i}(t) \Big) + c(t),$$

$k_0 = \prod_{i=0}^{m} \eta_i^{-\eta_i}$ *and η_i, $i = 0, \ldots, n$ are positive constants satisfying* $\sum_{i=1}^{m} \alpha_i \eta_i = 1$ *and* $\sum_{i=0}^{m} \eta_i = 1$, *then all solutions of (2) are oscillatory.*

Concept of oscillation for linear PDE

$$\Delta u + c(x)u = 0 \tag{4}$$

- Equation (4) is *oscillatory* if every solution has a zero on $\{x \in \mathbb{R}^n : \|x\| \geq a\}$ for each a.
- Equation (4) is *nodally oscillatory* if every solution has a nodal domain on $\{x \in \mathbb{R}^n : \|x\| \geq a\}$ for each a.
- Both definition are equivalent (Moss+Piepenbrink).

Concept of oscillation for half-linear PDE

$$\mathrm{div}\Big(\|\nabla u\|^{p-2}\,\nabla u\Big) + c(x)|u|^{p-2}u = 0 \tag{5}$$

- Essentialy the same approach to oscillation as in linear case
- The equivalence between two oscillations is open problem.

$$\operatorname{div}\left(A(x)\,\|\nabla y\|^{p-2}\,\nabla y\right) + \left\langle \vec{b}(x), \|\nabla y\|^{p-2}\,\nabla y \right\rangle + c(x)|y|^{p-2}y + \sum_{i=1}^{m} c_i(x)|y|^{p_i-2}y = e(x), \tag{E}$$

Detection of oscillation from ODE

Theorem B (O. Došlý (2001)). *Equation*

$$\operatorname{div}(\|\nabla u\|^{p-2}\nabla u) + c(x)|u|^{p-2}u = 0 \tag{6}$$

is oscillatory, if the ordinary differential equation

$$\left(r^{n-1}|u'|^{p-2}u'\right)' + r^{n-1}\left(\frac{1}{\omega_n r^{n-1}}\int_{S(r)} c(x)\,\mathrm{d}x\right)|u|^{p-2}u = 0 \tag{7}$$

is oscillatory. The number ω_n is the surface area of the unit sphere in $\mathbb{R}^n$.

J. Jaroš, T. Kusano and N. Yoshida proved independently similar result (for $A(x) = a(\|x\|)I$, $a(\cdot)$ differentiable).

Our aim

- Extend method used in Theorem A to (E). Derive a general result, like Theorem B.
- Derive a result which does depend on more general expression, than the mean value of $c(x)$ over spheres centered in the origin.
- Remove restrictions used by previous authors (for example Xu (2009) excluded the possibility $p_i > p$ for every i).

$$\operatorname{div}\left(A(x)\,\|\nabla y\|^{p-2}\,\nabla y\right) + \left\langle \vec{b}(x), \|\nabla y\|^{p-2}\,\nabla y \right\rangle + \boldsymbol{c(x)|y|^{p-2}y} + \sum_{i=1}^{m} \boldsymbol{c_i(x)|y|^{p_i-2}y} = \boldsymbol{e(x)}, \qquad \text{(E)}$$

Modus operandi

- Get rid of terms $\sum_{i=1}^{m} c_i(x)|y|^{p_i-2}y$ and $e(x)$ (join with $c(x)|y|^{p-2}y$) and convert the problem into
$$\operatorname{div}\left(A(x)\,\|\nabla y\|^{p-2}\,\nabla y\right) + \left\langle \vec{b}(x), \|\nabla y\|^{p-2}\,\nabla y \right\rangle + \boldsymbol{C(x)|y|^{p-2}y} = 0.$$
- Derive Riccati type inequality in n variables.
- Derive Riccati type inequality in 1 variable.
- Use this inequality as a tool which transforms results from ODE to PDE.

Using generalized AG inequality $\sum \alpha_i \geq \prod \left(\frac{\alpha_i}{\eta_i}\right)^{\eta_i}$, if $\alpha_i \geq 0$, $\eta_i > 0$ and $\sum \eta_i = 1$ we eliminate the right-hand side and terms with mixed powers.

Lemma 1. *Let either $y > 0$ and $e(x) \leq 0$ or $y < 0$ and $e(x) \geq 0$. Let $\eta_i > 0$ be numbers satisfying $\sum_{i=0}^{m} \eta_i = 1$ and $\eta_0 + \sum_{i=1}^{m} p_i \eta_i = p$ and let $c_i(x) \geq 0$ for every i. Then*

$$\frac{1}{|y|^{p-2}y}\left(-e(x) + \sum_{i=1}^{m} c_i(x)|y|^{p_i-2}y\right) \geq C_1(x),$$

where

$$C_1(x) := \left|\frac{e(x)}{\eta_0}\right|^{\eta_0} \prod_{i=1}^{m} \left(\frac{c_i(x)}{\eta_i}\right)^{\eta_i}. \tag{8}$$

Remark: The numbers η_i from Lemma 1 exist, if $p_i > p$ for some i.

Lemma 2. *Suppose $c_i(x) \geq 0$. Let $\eta_i > 0$ be numbers satisfying $\sum_{i=1}^{m} \eta_i = 1$ and $\sum_{i=1}^{m} p_i \eta_i = p$. Then*

$$\frac{1}{|y|^{p-2}y} \sum_{i=1}^{m} c_i(x)|y|^{p_i-2}y \geq C_2(x),$$

where

$$C_2(x) := \prod_{i=1}^{m} \left(\frac{c_i(x)}{\eta_i}\right)^{\eta_i} \tag{9}$$

Remark: The numbers η_i from Lemma 2 exist iff $p_i > p$ for some i and $p_j < p$ for some j.

Lemma 3. *Let y be a solution of (E) which does not have zero on Ω. Suppose that there exists a function $C(x)$ such that*

$$C(x) \leq c(x) + \sum_{i=1}^{m} c_i(x)|y|^{p_i - p} - \frac{e(x)}{|y|^{p-2}y}$$

Denote $\vec{w}(x) = A(x)\dfrac{\|\nabla y\|^{p-2}\nabla y}{|y|^{p-2}y}$. The function $\vec{w}(x)$ is well defined on Ω and satisfies the inequality

$$\operatorname{div}\vec{w} + (p-1)\Lambda(x)\,\|\vec{w}\|^q + \left\langle \vec{w}, A^{-1}(x)\vec{b}(x) \right\rangle + C(x) \leq 0 \tag{10}$$

where

$$\Lambda(x) = \begin{cases} \lambda_{\max}^{1-q}(x) & 1 < p \leq 2, \\ \lambda_{\min}\lambda_{\max}^{-q}(x) & p > 2. \end{cases} \tag{11}$$

Lemma 4. *Let (10) hold. Let $l > 1$, $l^* = \dfrac{l}{l-1}$ be two mutually conjugate numbers and $\alpha \in C^1(\Omega, \mathbb{R}^+)$ be a smooth function positive on Ω. Then*

$$\operatorname{div}(\alpha(x)\vec{w}) + (p-1)\frac{\Lambda(x)\alpha^{1-q}(x)}{l^*}\,\|\alpha(x)\vec{w}\|^q - \frac{l^{p-1}\alpha(x)}{p^p\Lambda^{p-1}(x)}\left\| A^{-1}(x)\vec{b}(x) - \frac{\nabla\alpha(x)}{\alpha(x)} \right\|^p + \alpha(x)C(x) \leq 0$$

holds on Ω. If $\left\| A^{-1}\vec{b} - \dfrac{\nabla\alpha}{\alpha} \right\| \equiv 0$ on Ω, then this inequality holds with $l^* = 1$.

Theorem 1. *Let the n-vector function $\vec{w}$ satisfy inequality*

$$\operatorname{div}\vec{w} + C_0(x) + (p-1)\Lambda_0(x)\,\|\vec{w}\|^q \le 0$$

on $\Omega(a,b)$. Denote $\widetilde{C}(r) = \int_{S(r)} C_0(x)\,\mathrm{d}\sigma$ and $\widetilde{R}(r) = \int_{S(r)} \Lambda_0^{1-p}\,\mathrm{d}\sigma$. Then the half-linear ordinary differential equation

$$\left(\widetilde{R}(r)|u'|^{p-2}u\right)' + \widetilde{C}(r)|u|^{p-2}u = 0, \qquad ' = \frac{\mathrm{d}}{\mathrm{d}r}$$

is disconjugate on $[a,b]$ and it possesses solution which has no zero on $[a,b]$.

Theorem 2. *Let $l > 1$. Let $l^* = 1$ if $\left\|\vec{b}\right\| \equiv 0$ and $l^* = \dfrac{l}{l-1}$ otherwise. Further, let $c_i(x) \ge 0$ for every i. Denote*

$$\widetilde{R}(r) = (l^*)^{p-1}\int_{S(r)} \Lambda^{1-p}(x)\,\mathrm{d}\sigma$$

and

$$\widetilde{C}(r) = \int_{S(r)} c(x) + C_1(x) - \frac{l^{p-1}}{p^p\Lambda^{p-1}(x)}\left\|A^{-1}(x)\vec{b}(x)\right\|^p\,\mathrm{d}\sigma\,,$$

where $\Lambda(x)$ is defined by (11) and $C_1(x)$ is defined by (8).
Suppose that the equation

$$\left(\widetilde{R}(r)|u'|^{p-2}u'\right)' + \widetilde{C}(r)|u|^{p-2}u = 0$$

has conjugate points on $[a,b]$.
If $e(x) \le 0$ on $\Omega(a,b)$, then equation (E) has no positive solution on $\Omega(a,b)$.
If $e(x) \ge 0$ on $\Omega(a,b)$, then equation (E) has no negative solution on $\Omega(a,b)$.

Theorem 3 (non-radial variant of Theorem 2). *Let $l > 1$ and let $\Omega \subset \Omega(a,b)$ be an open domain with piecewise smooth boundary such that* $\operatorname{meas}(\Omega \cap S(r)) \neq 0$ *for every $r \in [a,b]$. Let $c_i(x) \geq 0$ on Ω for every i and let $\alpha(x)$ be a function which is positive and continuously differentiable on Ω and vanishes on the boundary and outside Ω. Let $l^* = 1$ if $\left\| A^{-1}\vec{b} - \frac{\nabla\alpha}{\alpha} \right\| \equiv 0$ on Ω and $l^* = \frac{l}{l-1}$ otherwise. In the former case suppose also that the integral*

$$\int_{S(r)} \frac{\alpha(x)}{\Lambda^{p-1}(x)} \left\| A^{-1}(x)\vec{b}(x) - \frac{\nabla\alpha(x)}{\alpha(x)} \right\|^p \mathrm{d}\sigma$$

which may have singularity on $\partial\Omega$ if $\Omega \neq \Omega(a,b)$ is convergent for every $r \in [a,b]$. Denote

$$\widetilde{R}(r) = (l^*)^{p-1} \int_{S(r)} \alpha(x)\Lambda^{1-p}(x)\,\mathrm{d}\sigma$$

and

$$\widetilde{C}(r) = \int_{S(r)} \boldsymbol{\alpha(x)} \left(c(x) + C_1(x) - \frac{l^{p-1}}{p^p\Lambda^{p-1}(x)} \left\| A^{-1}(x)\vec{b}(x) - \frac{\nabla\alpha(x)}{\alpha(x)} \right\|^p \right) \mathrm{d}\sigma\,,$$

where $\Lambda(x)$ is defined by (11) and $C_1(x)$ is defined by (8) and suppose that equation

$$\left(\widetilde{R}(r)|u'|^{p-2}u' \right)' + \widetilde{C}(r)|u|^{p-2}u = 0$$

has conjugate points on $[a,b]$.
If $e(x) \leq 0$ on $\Omega(a,b)$, then equation (E) has no positive solution on $\Omega(a,b)$.
If $e(x) \geq 0$ on $\Omega(a,b)$, then equation (E) has no negative solution on $\Omega(a,b)$.

Theorem 4. *Let* l, Ω, $\alpha(x)$, $\Lambda(x)$ *and* $\widetilde{R}(r)$ *be defined as in Theorem 3 and let* $c_i(x) \geq 0$ *and* $\boldsymbol{e(x) \equiv 0}$ *on* $\Omega(a,b)$. *Denote*

$$\widetilde{C}(r) = \int_{S(r)} \alpha(x) \left(c(x) + C_2(x) - \frac{l^{p-1}}{p^p \Lambda^{p-1}(x)} \left\| A^{-1}(x)\vec{b}(x) - \frac{\nabla \alpha(x)}{\alpha(x)} \right\|^p \right) \mathrm{d}\sigma ,$$

where $C_2(x)$ *is defined by* (9). *If the equation*

$$\left(\widetilde{R}(r)|u'|^{p-2}u' \right)' + \widetilde{C}(r)|u|^{p-2}u = 0$$

has conjugate points on $[a,b]$, *then every solution of equation* (E) *has zero on* $\Omega(a,b)$.

Similar theorems can be derived also for estimates of terms with mixed powers based on different methods than AG inequality (see R. M., Nonlinear Analysis TMA 73 (2010)).