11-22-1993

Problems on the Heat Equation and the Wave Equation.

1. Solve the heat equation

$$u_t = 16u_{xx}, \qquad 0 < x < 3, \quad t > 0$$

$$u(0,t) = 0 = u(3,t), \qquad t > 0$$

$$u(x,0) = 6\sin(3\pi x) - 5\sin(\pi x), \qquad 0 \le x \le 3$$

2. Solve the heat equation

$$u_t = 16u_{xx}, \qquad 0 < x < 3, \quad t > 0$$

$$u(0,t) = 0 = u(3,t), \qquad t > 0$$

$$u(x,0) = \sin x, \qquad 0 \le x \le 3$$

3. Solve the heat equation

$$u_t = 4u_{xx}, \qquad 0 < x < 1, \quad t > 0$$

$$u_x(0,t) = u_x(1,t) = 0, \qquad t > 0$$

$$u(x,0) = x^2, \qquad 0 \le x \le 1$$

4. Solve the wave equation

$$u_{tt} = u_{xx}, \qquad 0 < x < 3, \quad t > 0$$

$$u(0,t) = u(3,t) = 0, \qquad t > 0$$

$$u(x,0) = \begin{cases} x & \text{if } 0 < x < 1 \\ 0 & \text{if } 1 \le x \le 2 \\ x-3 & \text{if } 2 < x < 3 \end{cases}$$

$$u_t(x,0) = \begin{cases} -1 & \text{if } 0 < x < 1 \\ 0 & \text{if } 1 \le x \le 2 \\ 1 & \text{if } 2 < x < 3 \end{cases}$$

Solutions to the Problems on the Heat Equation and the Wave Equation.

1. Solve the heat equation

$$u_t = 16u_{xx}, \qquad 0 < x < 3, \quad t > 0$$

$$u(0,t) = 0 = u(3,t), \qquad t > 0$$

$$u(x,0) = 6\sin(3\pi x) - 5\sin(\pi x), \qquad 0 \le x \le 3$$

The solution has the form

$$u(x,t) = \sum_{n=1}^{\infty} b_n \exp\left(-\frac{\pi^2 a^2 n^2 t}{L^2}\right) \sin\left(\frac{\pi n x}{L}\right).$$

Since $u(x,0)$ is a sum of sines whose periods are integer fractions of $L = 3$, I can find the coefficients by comparison.

$$6\sin(3\pi x) - 5\sin(\pi x) = u(x,0) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{\pi n x}{3}\right).$$

$6\sin(3\pi x) = b_n \sin\left(\frac{\pi n x}{3}\right)$ gives $n = 9$ and $b_9 = 6$.
$-5\sin(\pi x) = b_n \sin\left(\frac{\pi n x}{3}\right)$ gives $n = 3$ and $b_3 = -5$.
The solution is

$$u(x,t) = -5\exp\left(-\frac{16\pi^2 3^2 t}{3^2}\right)\sin(\pi x) + 6\exp\left(-\frac{16\pi^2 9^2 t}{3^2}\right)\sin(3\pi x) =$$

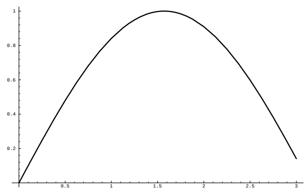
$$-5\exp(-16\pi^2 t)\sin(\pi x) + 6\exp(-144\pi^2 t)\sin(3\pi x).\ \square$$

2. Solve the heat equation

$$\begin{aligned} u_t &= 16u_{xx}, \qquad 0 < x < 3, \quad t > 0 \\ u(0,t) &= 0 = u(3,t), \qquad t > 0 \\ u(x,0) &= \sin x, \qquad 0 \le x \le 3 \end{aligned}$$

Notice that this problem looks like the first problem, down to the sine function as the initial distribution. However, you *can't* do this one by comparing terms. Why? Because the solution to this heat equation will have sine terms of the form $\sin\frac{\pi n x}{3}$, where n is an integer. For no *integer* n will such a term equal $\sin x$ — how can you cancel the π?

To put it another way, the period of $\sin x$ is not commensurable with the length of the interval, as you can see from the graph.



The solution is

$$u(x,t) = \sum_{n=1}^{\infty} b_n \exp\left(-4\pi^2 n^2 t\right)\sin\left(\frac{\pi n x}{3}\right).$$

The b_n's are the coefficients in the Fourier sine series for $\sin x$. This sounds funny, but the point is that $\sin x$ and $\sin\left(\frac{\pi n x}{3}\right)$ are sines of *different fundamental periods.*

Now

$$b_n = \frac{2}{3}\int_0^3 \sin x \sin\left(\frac{\pi n x}{3}\right)\,dx.$$

Use the identity

$$\sin a \sin b = \frac{1}{2}\left[\cos(a-b) - \cos(a+b)\right].$$

You can derive this from the angle addition and subtraction formulas for cosine, or from the complex exponential identity $e^{(a+b)i} = e^{ai}e^{bi}$ (write both sides in terms of cosines and sines, multiply out, and compare real and imaginary parts). Alternatively, you can do the integral as is by repeated integration by parts.

Anyway, using the identity,

$$b_n = \frac{1}{3}\int_0^3 \left(\cos\left(1 - \frac{\pi n}{3}\right)x - \cos\left(1 + \frac{\pi n}{3}\right)x\right)\,dx =$$

$$\frac{1}{3}\left[\frac{1}{1-\dfrac{\pi n}{3}} \sin\left(1-\frac{\pi n}{3}\right)x - \frac{1}{1+\dfrac{\pi n}{3}} \sin\left(1+\frac{\pi n}{3}\right)x\right]_0^3 = \frac{1}{3-\pi n}\sin(3-\pi n) - \frac{1}{3+\pi n}\sin(3+\pi n).$$

The solution is

$$u(x,t) = \sum_{n=1}^{\infty}\left(\frac{1}{3-\pi n}\sin(3-\pi n) - \frac{1}{3+\pi n}\sin(3+\pi n)\right)\exp\left(-4\pi^2 n^2 t\right)\sin\left(\frac{\pi n x}{3}\right). \ \square$$

3. Solve the heat equation

$$\begin{gathered} u_t = 4u_{xx}, \qquad 0 < x < 1, \quad t > 0 \\ u_x(0,t) = u_x(1,t) = 0, \qquad t > 0 \\ u(x,0) = x^2, \qquad 0 \le x \le 1 \end{gathered}$$

The equations describe heat conduction in a bar with insulated ends. The constants are $L = 1$ and $a^2 = 4$. The solution is

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \exp\left(-4\pi^2 n^2 t\right)\cos \pi n x.$$

The a_n's are the coefficients in the Fourier cosine series for $u(x,0)$. All you have to do is to compute the a_n's.

First,

$$a_0 = \frac{2}{1}\int_0^1 x^2\,dx = \frac{2}{3}.$$

Next,

$$a_n = \frac{2}{1}\int_0^1 x^2 \cos \pi n x\,dx = 2\left[\frac{1}{\pi}kx^2 \sin \pi n x + \frac{2}{\pi^2 n^2}x\cos \pi n x - \frac{2}{\pi^3}n^3 \sin \pi n x\right]_0^1 = \frac{4}{\pi^2 n^2}\cos \pi n.$$

Here's the table for the integration by parts:

$$\begin{array}{ccc} & \dfrac{d}{dx} & \int dx \\ + & x^2 & \cos \pi n x \\ & & \searrow \\ - & 2x & \dfrac{1}{\pi n}\sin \pi n x \\ & & \searrow \\ + & 2 & -\dfrac{1}{\pi^2 n^2}\cos \pi n x \\ & & \searrow \\ - & 0 & -\dfrac{1}{\pi^3 n^3}\sin \pi n x \end{array}$$

The solution is

$$u(x,t) = \frac{1}{3} + \sum_{n=1}^{\infty}\frac{4}{\pi^2 n^2}\cos(\pi n)\exp\left(-4\pi^2 n^2 t\right)\cos \pi n x. \ \square$$

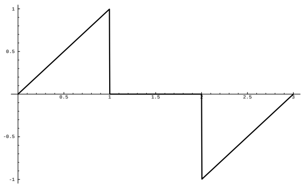
4. Solve the wave equation

$$\begin{gathered} u_{tt} = u_{xx}, \qquad 0 < x < 3, \quad t > 0 \\ u(0,t) = u(3,t) = 0, \qquad t > 0 \end{gathered}$$

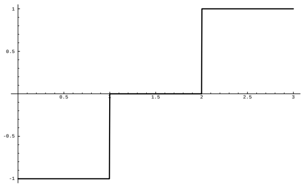
$$u(x,0) = \begin{cases} x & \text{if } 0 < x < 1 \\ 0 & \text{if } 1 \le x \le 2 \\ x-3 & \text{if } 2 < x < 3 \end{cases}$$

$$u_t(x,0) = \begin{cases} -1 & \text{if } 0 < x < 1 \\ 0 & \text{if } 1 \le x \le 2 \\ 1 & \text{if } 2 < x < 3 \end{cases}$$

Here is the initial displacement:



Here is the initial velocity:



The solution is

$$u(x,t) = \sum_{n=1}^{\infty} \sin \frac{\pi n x}{L} \left(b_n \cos \frac{\pi n a t}{L} + k_n \sin \frac{\pi n a t}{L} \right).$$

b_n is the n-th Fourier sine coefficient of the initial displacement. $\dfrac{\pi n a}{L} k_n$ is the n-th Fourier sine coefficient of the initial velocity. (The $\dfrac{\pi n a}{L}$ factor comes from differentiating $u(x,t)$ with respect to t, then plugging in $t = 0$.)

First,

$$b_n = \frac{2}{3}\left(\int_0^1 x \sin \frac{\pi n x}{3}\, dx + \int_2^3 (x-3) \sin \frac{\pi n x}{3}\, dx\right) =$$

$$\frac{2}{3}\left(\left[-\frac{3}{\pi n} x \cos \frac{\pi n x}{3} + \frac{9}{\pi^2 n^2} \sin \frac{\pi n x}{3}\right]_0^1 + \left[-\frac{3}{\pi n}(x-3) \cos \frac{\pi n x}{3} + \frac{9}{\pi^2 n^2} \sin \frac{\pi n x}{3}\right]_2^3\right) =$$

$$2\left[-\frac{1}{\pi n}\left(\cos \frac{2\pi n}{3} + \cos \frac{\pi n}{3}\right) + \frac{3}{\pi^2 n^2}\left(\sin \frac{\pi n}{3} - \sin \frac{2\pi n}{3}\right)\right].$$

(Yow!) Next, since $a = 1$,

$$\frac{\pi n}{3} k_n = \frac{2}{3} \left(\int_0^1 (-1) \sin \frac{\pi n x}{3}\, dx + \int_2^3 (1) \sin \frac{\pi n x}{3}\, dx \right) =$$

$$\frac{2}{\pi n} \left[\left(\cos \frac{\pi n}{3} - 1 \right) - \left(\cos \pi n - \cos \frac{2\pi n}{3} \right) \right].$$

So

$$k_n = \frac{6}{\pi^2 n^2} \left[\left(\cos \frac{\pi n}{3} - 1 \right) - \left(\cos \pi n - \cos \frac{2\pi n}{3} \right) \right].$$

The solution is

$$u(x,t) = \sum_{n=1}^{\infty} \sin \frac{\pi n x}{3} \left(b_n \cos \frac{\pi n t}{3} + k_n \sin \frac{\pi n t}{3} \right)$$

with b_n and k_n as above. □