



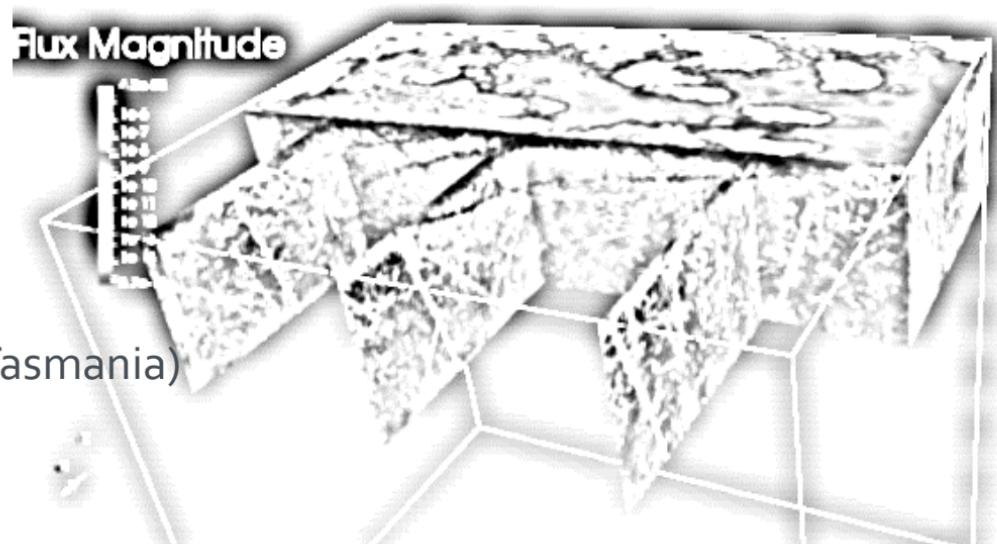
Geophysical applications beyond (but in connection to) Elmer/Ice

Thomas Zwinger
ElmerTeam
CSC – IT Center for Science, Finland

Elmer FEM webinar
15 Dec 2021

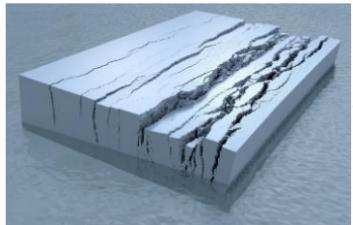
With contributions by

- **Elmer Team:** Mika Malinen, Juha Ruokolainen and Peter Råback
- **Denis Cohen** (NM Tech, CoSci)
- **Juha Hartikainen** (Univ. Tampere)
- **Matt King** (Univ. Tasmania)
- **Grace Nield** (Univ. Durham + Univ. Tasmania)



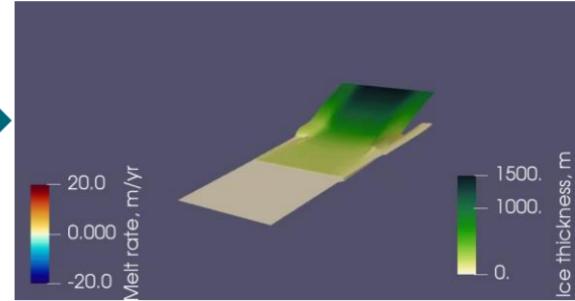
Beyond Elmer/Ice

Coupler to DEM (HiDEM)

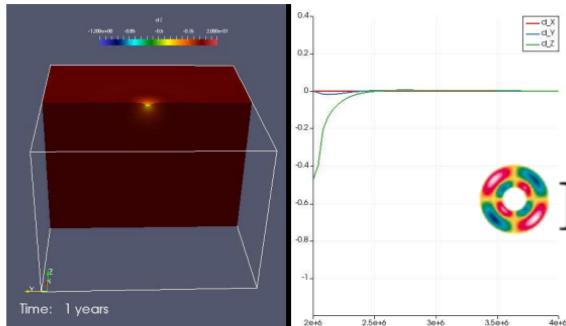


The image features the Elmer ICE logo. On the left, the word "elmer" is written in a yellow, rounded font, and "ICE" is in a blue, blocky font. The letter "e" in "elmer" is replaced by a colorful, abstract swirl of red, orange, yellow, green, and blue. To the right of the text is a large, irregularly shaped mesh representing an ice crystal, composed of many small triangles. At the bottom of the image are two large, solid teal arrows pointing upwards, one on each side.

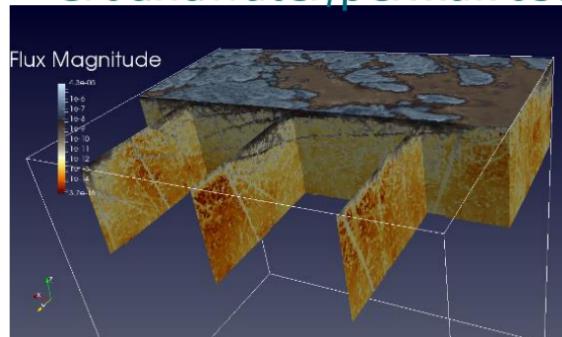
Ice-ocean coupler (FISOC)



Visco-elastic earth model



Groundwater/permafrost model

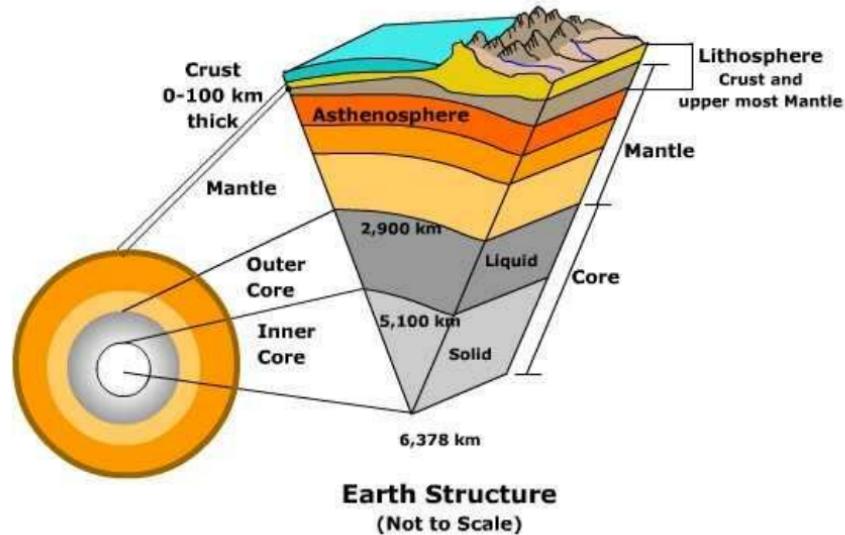


Viscoelastic earth model



Structure of Earth

- Lithosphere:
 - Earth's crust only in average 20 km thick (~ brittle, elastic plate)
 - Minerals; $\rho \approx 2900 \text{ kg m}^{-3}$
 - Asthenosphere (mineral upper part of mantle):
 - $c; \eta \approx 10^{19} \dots 10^{21} \text{ Pa s}$
- Mantle:
 - All in all 2900 km thick; mineral
 - At bottom $\rho \approx 5700 \text{ kg m}^{-3}$
- Core (metallic):
 - $\rho \approx 9400 \rightarrow 13500 \text{ kg m}^{-3}$
 - Outer, 3200 km thick liquid core
 - Inner, 1200 km thick solid core



Picture taken from
http://www.sfu.ca/geog/geog351fall06/group06/Image/Earthquake/EQ%20Picture%20earth_structure.jpg

Structure of Earth

- Earth has a layer type of structure
- Determined by seismic measurements
- PREM (Preliminary

Physics of the Earth and Planetary Interiors, 25 (1981) 297–356
Elsevier Scientific Publishing Company, Amsterdam – Printed in The Netherlands

Preliminary reference Earth model *

Adam M. Dziewonski¹ and Don L. Anderson²

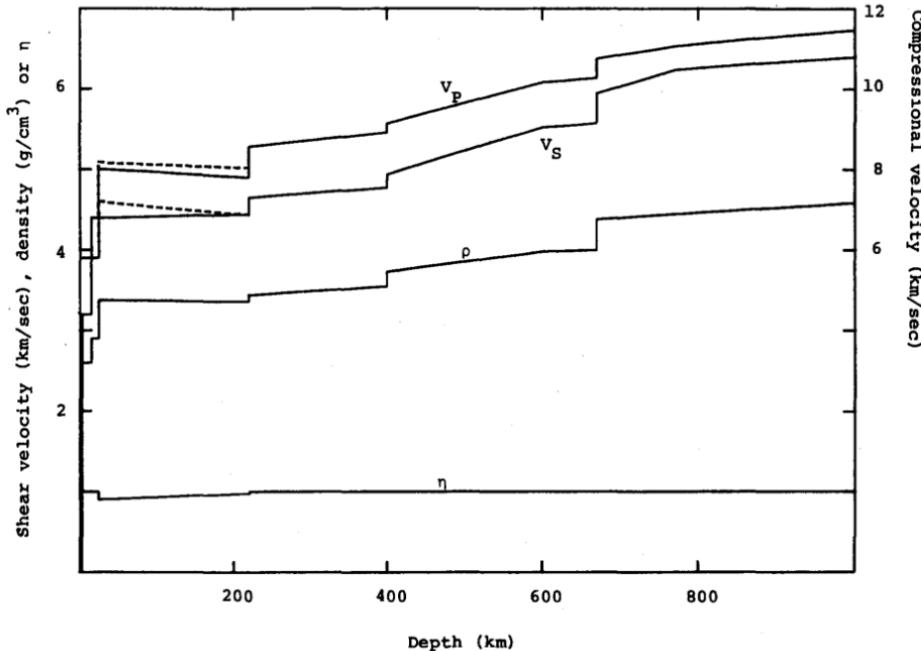
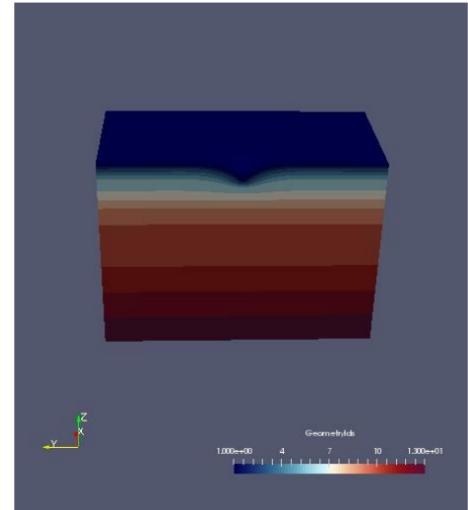


Fig. 7. Upper mantle velocities, density and anisotropic parameter η in PREM. The dashed lines are the horizontal components of velocity. The solid curves are η , ρ and the vertical, or radial, components of velocity.

Structure of Earth

Layer	Layer top (radius, km)	Layer base (radius, km)	Thickness (km)	Viscosity	Density	Young's Modulus	Poisson's Ratio	Gravitational Acceleration
Lithosphere	6371	6336	35	1x10^44	3196	1.8148E+11	0.4	9.7852
Lithosphere	6336	6301	35	1x10^44	3196	1.8148E+11	0.4	9.7852
Lithosphere	6301	6251	50	1x10^44	3196	1.8148E+11	0.4	9.7852
Upper Mantle	6251	6201	50	1x10^18	3439	2.1901E+11	0.4	9.8367
Upper Mantle	6201	6141	60	1x10^18	3439	2.1901E+11	0.4	9.8367
Upper Mantle	6141	5971	170	1x10^18	3439	2.1901E+11	0.4	9.8367
Upper Mantle	5971	5835	136	1x10^18	3882	3.2393E+11	0.4	9.9349
Upper Mantle	5835	5701	134	1x10^18	3882	3.2393E+11	0.4	9.9349
Lower Mantle	5701	5450	251	1x10^22	4527	5.3663E+11	0.4	9.9799
Lower Mantle	5450	4770	680	1x10^22	4527	5.3663E+11	0.4	9.9799
Lower Mantle	4770	4340	430	1x10^22	5074	7.2010E+11	0.4	9.9108
Lower Mantle	4340	3910	430	1x10^22	5074	7.2010E+11	0.4	9.9108
Lower Mantle	3910	3480	430	1x10^22	5074	7.2010E+11	0.4	9.9108

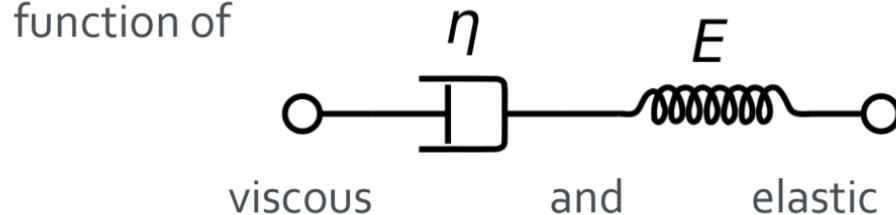


Non-self gravitating Flat-earth model

Implementation in Elmer

- Standard FE *linear elasticity*: $\nabla \cdot \tau = 0$
- Elastic rheology: stress as a function of reversible deformation
- Visco-elastic – Maxwell rheology :

(partly non-reversible) deformation as a function of



contribution

By Pekaje at English Wikipedia - Transferred from en.wikipedia to Commons., Public Domain

Implementation in Elmer

- Introduction of visco-elastic stress (Wu 2004)

$$\frac{\partial \boldsymbol{\tau}}{\partial t} = \frac{\partial \boldsymbol{\tau}_0}{\partial t} + \frac{\mu}{\eta} (\boldsymbol{\tau} - \Pi \mathbf{1})$$

$$\boldsymbol{\tau}_0 = \Pi \mathbf{1} + 2\mu \boldsymbol{\epsilon}$$

(Maxwell time)⁻¹

- At the same time we introduce pressure Π to enable incompressibility
- Additional term accounting for restoring force by specific weight gradient (aka. pre-stress advection)

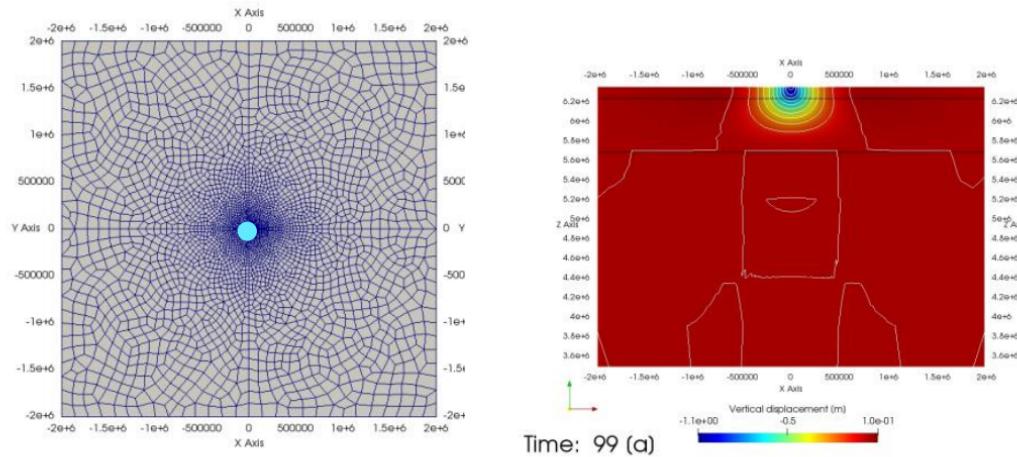
$$\nabla \cdot \boldsymbol{\tau} - \rho g \nabla (e_z \cdot \mathbf{d}) = 0$$

- This is not standard in commercial FE packages, hence needs to be “cheated” around by putting jump-conditions on inter-layer boundaries (Winkler foundations)
- In Elmer we can include this, which introduces the right boundary condition naturally from the third term of the weak formulation

$$\int_{\Omega} \boldsymbol{\tau}(\mathbf{u}) \cdot \boldsymbol{\epsilon}(\mathbf{v}) dV - \oint_{\partial\Omega} (\boldsymbol{\tau}(\mathbf{u}) \cdot \mathbf{n}) \cdot \mathbf{v} dA - \int_{\Omega} \rho g \nabla (e_z \cdot \mathbf{u}) \cdot \mathbf{v} dV = 0.$$

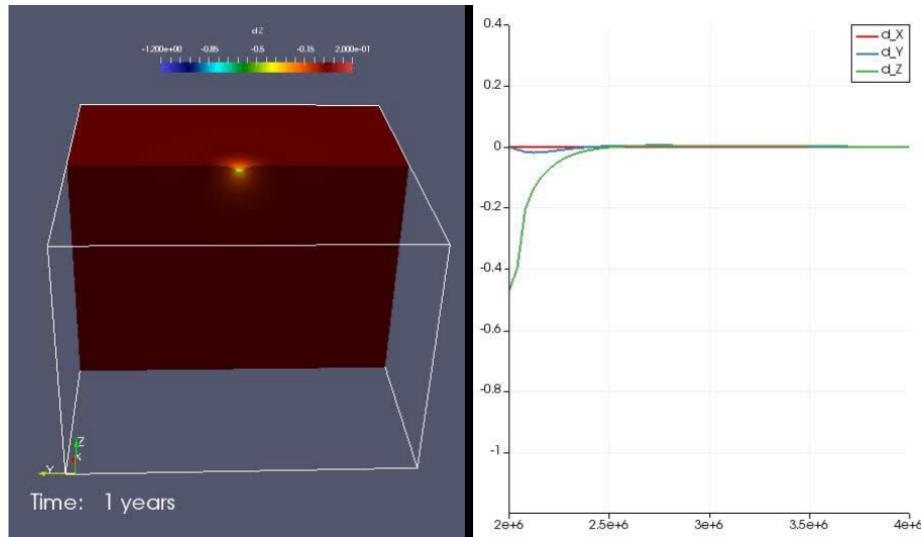
GIA benchmark model

- Total width 4000km (2000km each side of the ice load centre)
- Depth – surface to core (6371 – 3480km)
- Load:
 - Disc radius: 50km (diameter 100km)
 - Disc thickness: 100m
 - Ice density: 917 kg/m³
 - Loading 100 years, unloading 100 years

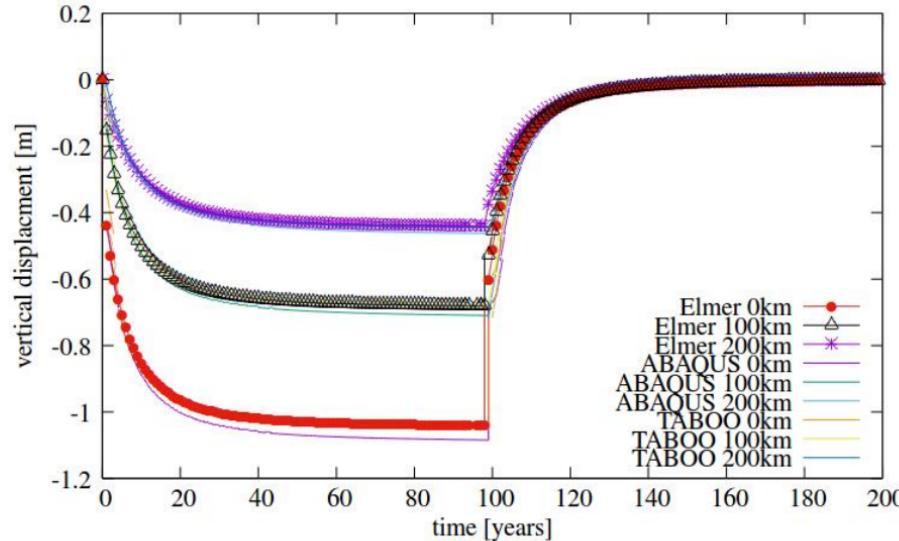


Zwinger, T., Nield, G. A., Ruokolainen, J., and King, M. A., 2020. *A new open-source viscoelastic solid earth deformation module implemented in Elmer (v8.4)*, Geosci. Model Dev., 14, 1155–1164, doi:10.5194/gmd-13-1155-2020

GIA benchmark model



GIA benchmark model – comparison to ABAQUS and TABOO

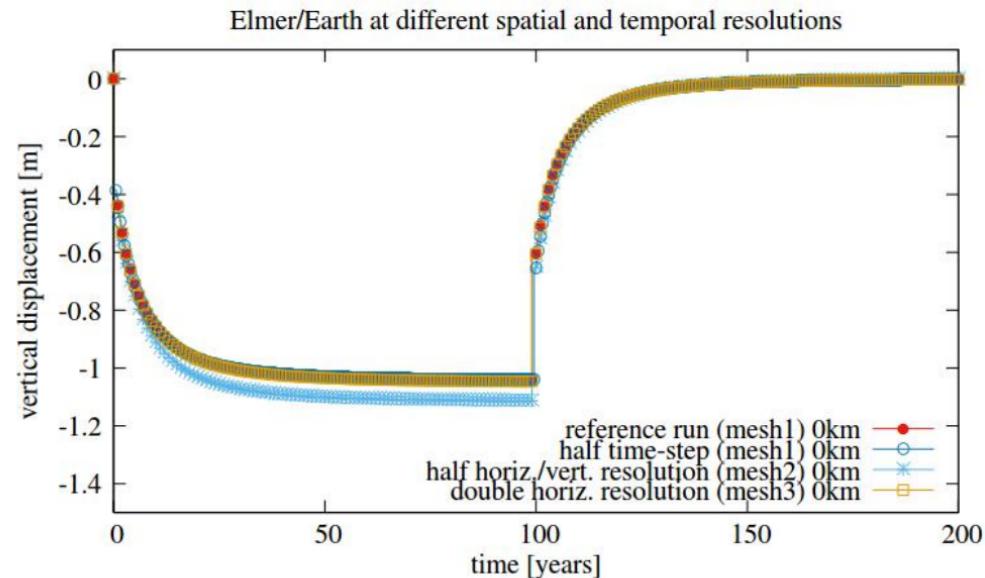


- ABAQUS: FEM model run at constant 25 km horizontal resolution
- TABOO: post-glacial rebound calculator (Spada et al., 2003) using classical visco-elastic normal mode method.
 - Incompressible, non-rotating, self-gravitational
 - Spherical harmonics degree 2048 (equivalent to approximately 10km)

GIA benchmark model – accuracy and performance

name	elements	min hor. res.
mesh 1	80k	~10 km
mesh 2 (half)	40k	~25 km
mesh 3 (double)	150k	~5 km

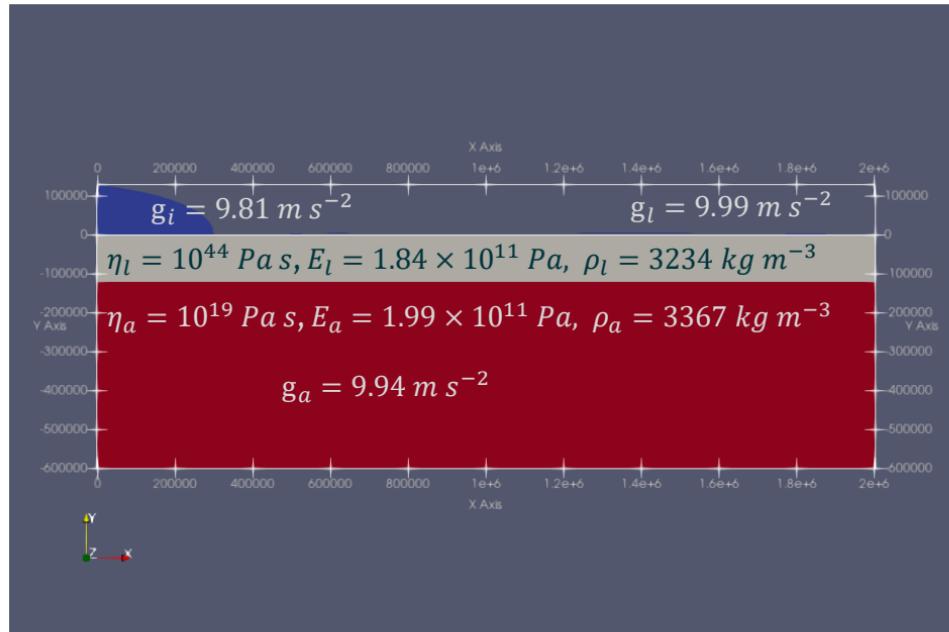
name	partitions	time [s]
mesh 1	16	21 200
mesh 1 (strong)	32	9600 (2.2 speedup)
mesh 3 (weak)	64	15800 (0.61 scaled speedup)



Simple 2D/2layer testcase in source code:
[elmerfem/fem/tests/elmer_earth_2d_2layer/](https://github.com/ElmerFEM/elmerfem/blob/master/fem/tests/elmer_earth_2d_2layer/)

Coupled to ice sheet

- Imposing ice sheet with a Bueler profile
- 5 kyr advance from 0-300km (1500 m thickness)
- 5 kyr retreat
- 2 layer model (crust 12 km + mantle 600 km)



Coupled to ice sheet

```
!----- MATERIALS -----
!
Material 1
  Name = "Ice Material"
  Density = Real #rhoi
End
! Lithosphere
Material 2
  Density = #rhob
  Damping = Real 0.0
  Youngs Modulus = #ymodl
  ! supper high viscosity, hence,
  ! Maxwell time is such that it acts elastic
  Viscosity = #viscl
  Poisson Ratio = Real 0.49 !not needed if incompressible
End
! Upper Mantle 1
Material 3
  Density = #rhoa
  Damping = Real 0.0
  Youngs Modulus = #ymoda
  Viscosity =#visca
  Poisson Ratio = Real 0.49 !not needed if incompressible
End
```

$$\frac{\partial \boldsymbol{\tau}}{\partial t} = \frac{\partial \boldsymbol{\tau}_0}{\partial t} + \frac{\mu}{\eta} (\boldsymbol{\tau} - \Pi \mathbf{1})$$

```
!----- BODY FORCES -----
!
Body Force 1
  Name = "Ice Bodyforce"
  Flow BodyForce 1 = Real 0
  Flow BodyForce 2 = Real #-gravity
End
! Lithosphere
Body Force 2
  Stress BodyForce 1 = 0.0
  Stress BodyForce 2 = 0.0
  Gravitational Prestress Advection = Logical True
  GPA Coeff = Real # rhob * gravb
End
!Upper Mantle 1
Body Force 3
  Stress BodyForce 1 = 0.0
  Stress BodyForce 2 = 0.0
  Gravitational Prestress Advection = Logical True
  GPA Coeff = Real # rhoa * grava
```

$$-\rho g \nabla (e_z \cdot d)$$

Coupled to ice sheet

```

=====
! // Visco-elastic solver //
=====

Solver 4
  Equation = "Elasticity Analysis"
  Procedure = "StressSolve" "StressSolver"

  Displace Mesh = Logical True ! physically deform the mesh?

  Calculate Stresses = Logical True ! outputs elastic stresses

  ! 2D: 2 deformation 1 pressure (as incompressible)
  Variable = String "t[d:2 p:1]"

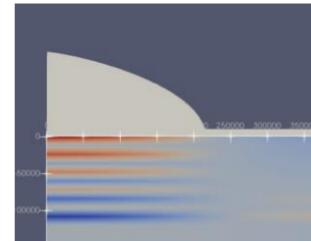
  ! if using p:1 and bubble, then either b:3 or b:6
  ! Element = "p:1 b:3"
  ! best to use p:2 for deofrmation (pressure will be p:1)
  Element = "p:2"

  ! Visco-elastic computation if Ture
  Maxwell material = Logical True
  Incompressible = Logical True
  Time Derivative Order = 1

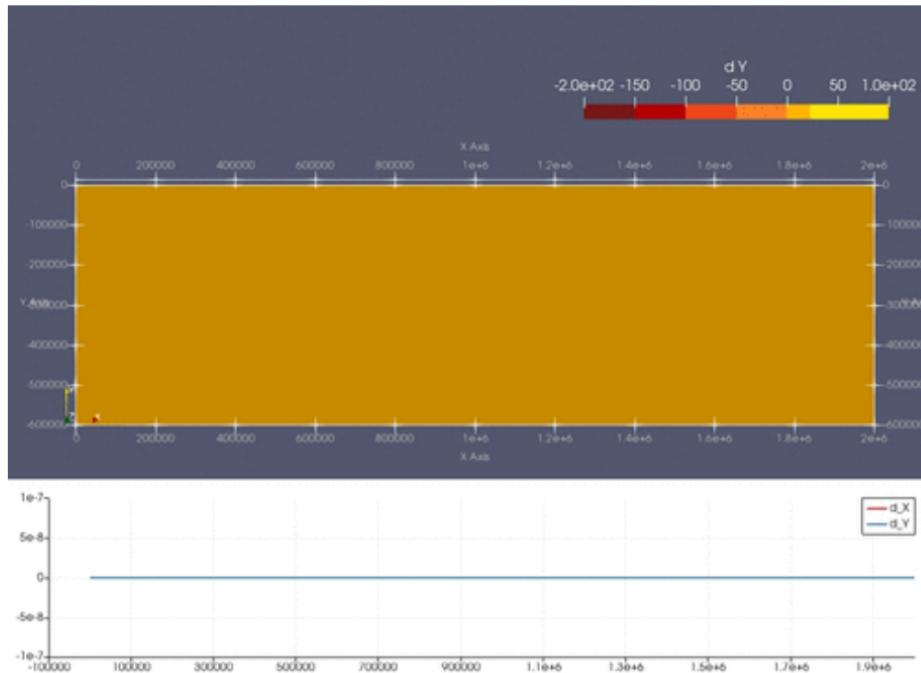
  ! Numerical settings (here direct Solver)
  Linear System Solver = Direct
  Linear System Direct Method = MUMPS
  Steady State Convergence Tolerance= 1.0e-5
End

```

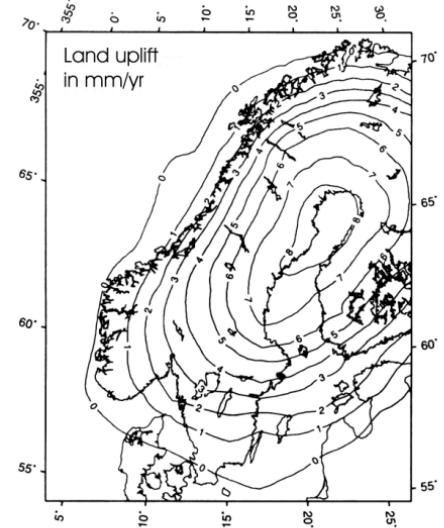
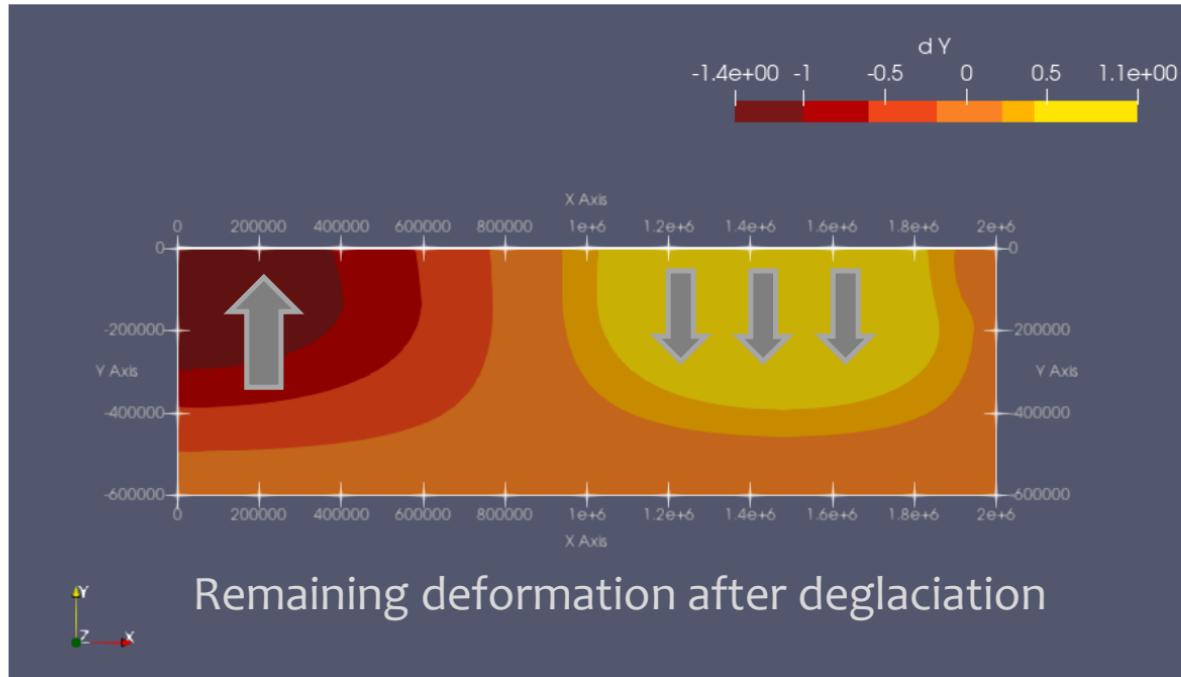
- Add-on functionality to existing linear elasticity solver (StressSolve)
- Incompressibility expects deformations + pressure DOF
- Best strategy for stabilization: p:2 (p:1)



Coupled to ice sheet

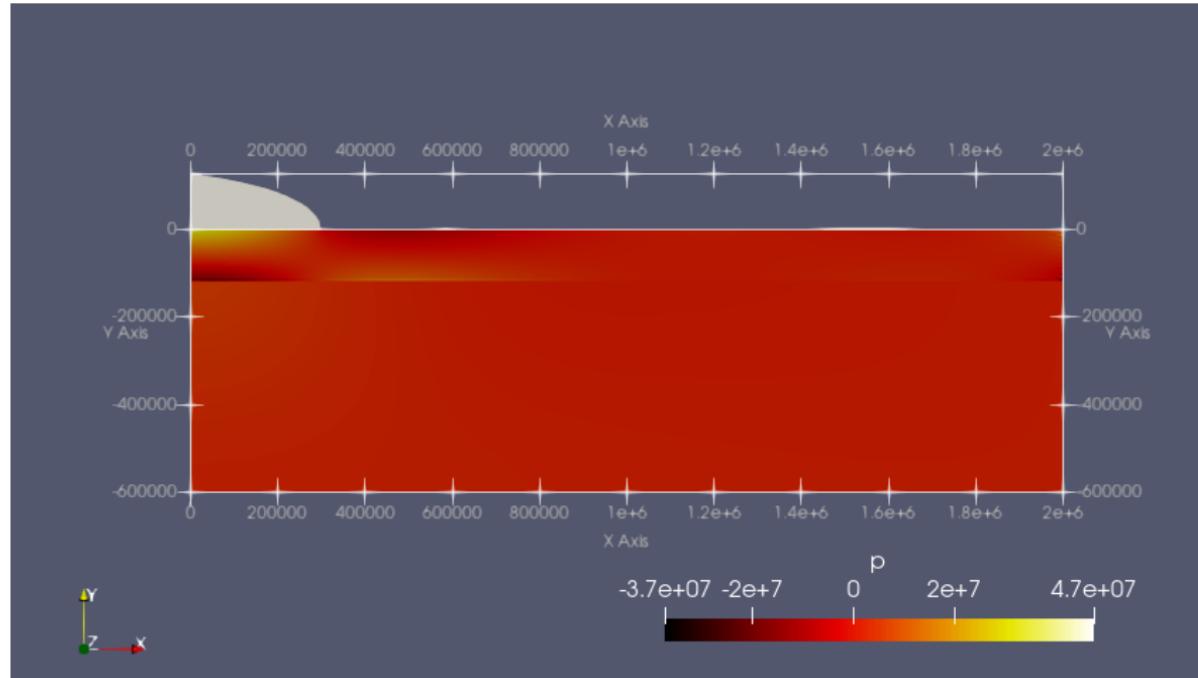


Coupled to ice sheet

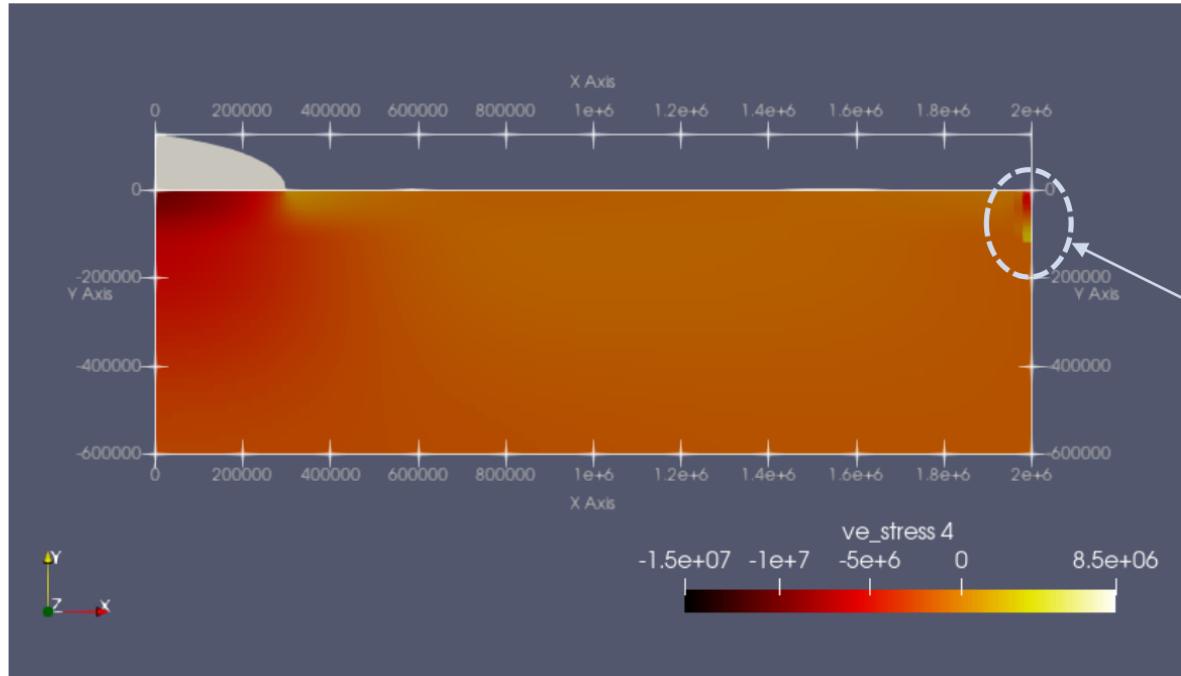


From Thoma
and Wolf, 1999

Coupled to ice sheet

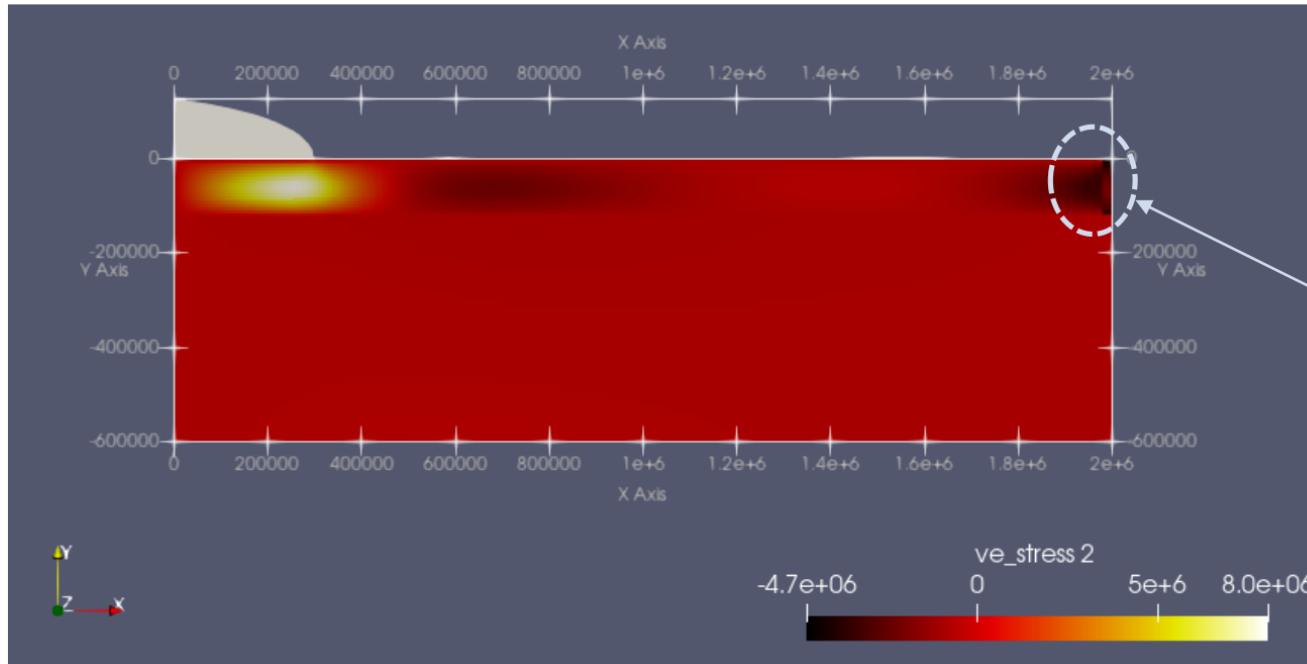


Coupled to ice sheet



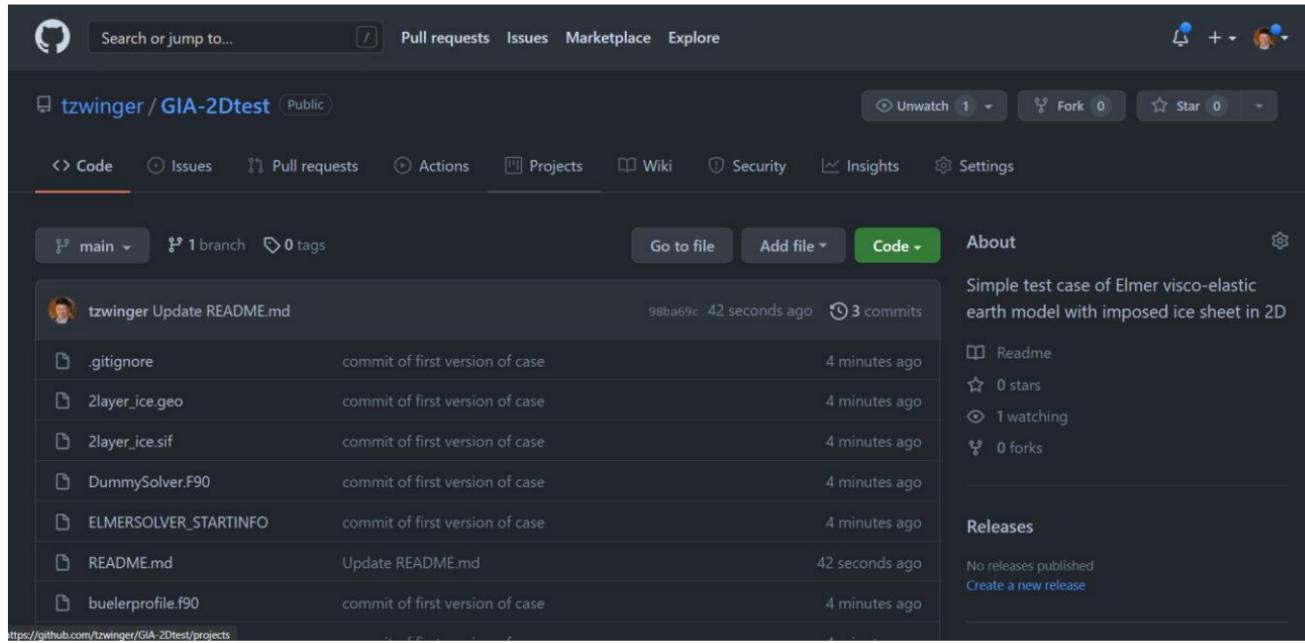
Far-field
condition too
close

Coupled to ice sheet



Far-field
condition too
close

Testcase to be found under GitHub: tzwinger/GIA-2Dtest



The screenshot shows a GitHub repository page for `tzwinger / GIA-2Dtest`. The repository is public and contains one branch and no tags. The `Code` tab is selected. The commit history shows the following changes:

File	Description	Time
<code>README.md</code>	commit of first version of case	4 minutes ago
<code>.gitignore</code>	commit of first version of case	4 minutes ago
<code>2layer_ice.geo</code>	commit of first version of case	4 minutes ago
<code>2layer_ice.sif</code>	commit of first version of case	4 minutes ago
<code>DummySolver.F90</code>	commit of first version of case	4 minutes ago
<code>ELMERSOLVER_STARTINFO</code>	commit of first version of case	4 minutes ago
<code>README.md</code>	Update <code>README.md</code>	42 seconds ago
<code>buelerprofile.f90</code>	commit of first version of case	4 minutes ago

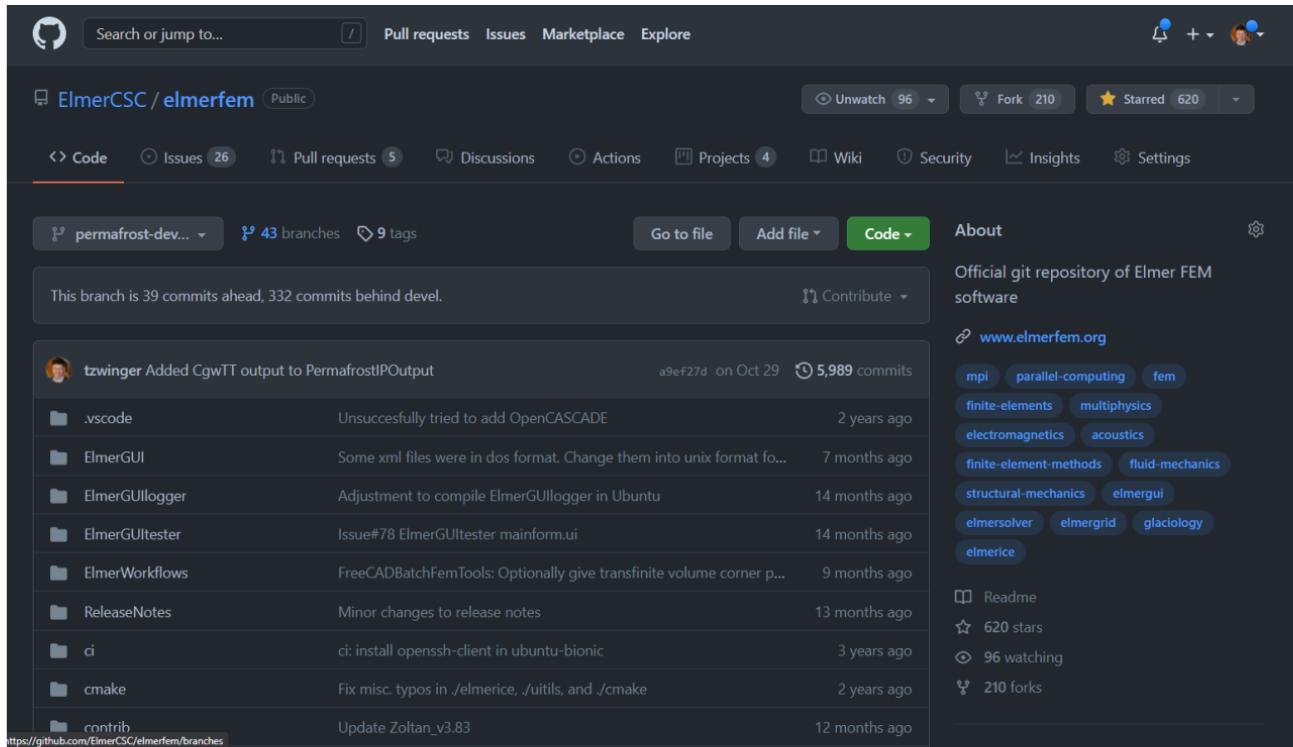
The repository has 3 commits, 0 stars, 1 watching, and 0 forks. The `About` section describes it as a simple test case of Elmer visco-elastic earth model with imposed ice sheet in 2D.

Groundwater-permafrost model



Groundwater-permafrost model

- This is still under heavy development
- Currently, special branch
permafrost-devel in the GitHub repository is assigned to it

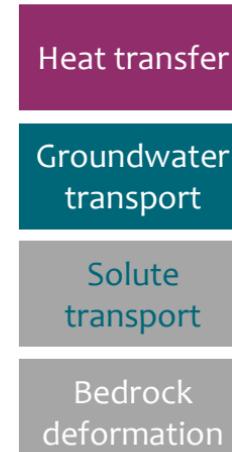


The screenshot shows the GitHub repository page for ElmerCSC/elmerfem. The repository is public and has 96 watchers, 210 forks, and 620 starred users. It contains 43 branches and 9 tags. The current branch is 'permafrost-devel'. A message indicates it is 39 commits ahead of '332' and 332 commits behind 'devel'. The repository has 5,989 commits. The main commit listed is by tzwinger, adding CgwTT output to PermafrostPOOutput. Other commits are listed for various files like .vscode, ElmerGUI, ElmerGULogger, ElmerGULTester, ElmerWorkflows, ReleaseNotes, ci, cmake, and contrib. The repository page also includes links to Readme, stars, watching, and forks.

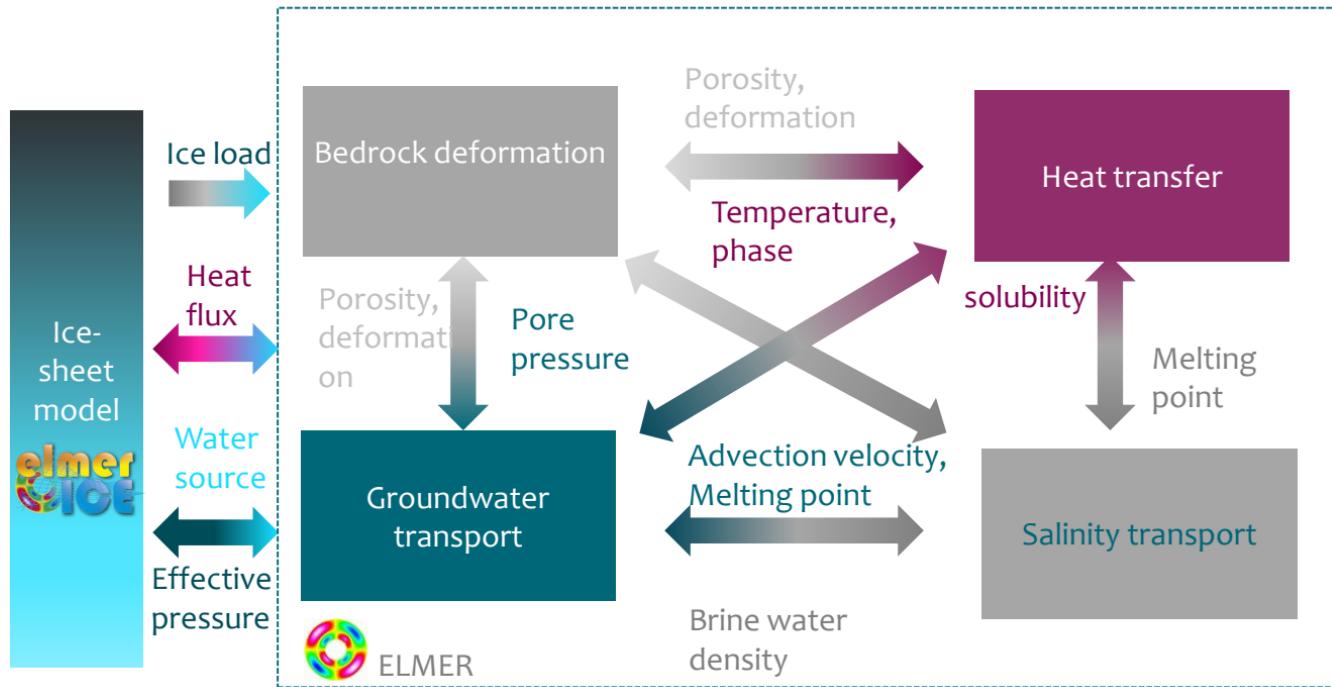
<https://github.com/ElmerCSC/elmerfem/branches>

Permafrost model

- Implemented in the open-source software **Elmer**
 - Finite Element Method (standard Galerkin, plan for mixed elem.)
 - Easily coupled with **Elmer/Ice** ice-sheet simulation
 - Extra module with about 7000 lines of code
 - Main motivation: MPI/OpenMP parallelism (up to 1000's of cores)
- Each physical problem implemented in separate module
 - Fixed-point iterations on all other variables
 - Assumption of saturated aquifer
- Residual free bubbles for stabilization in case of advection
 - Potential to vectorise algorithm
- Dependent material parameters evaluated at IP's
 - Needed for accuracy
- p-elements of different orders



Permafrost model



Permafrost model

- Advection/Diffusion(/Reaction) type of equation

Groundwater transport

Advection Diffusion

$$C_{\text{gw}}^{pp} \left(\frac{\partial p}{\partial t} + \mathbf{v}_* \cdot \text{grad } p \right) + \text{div} (\varrho_{\text{gw}} \mathbf{J}_{\text{gw}}^D) + C_{\text{gw}}^{pT} \left(\frac{\partial T}{\partial t} + \mathbf{v}_* \cdot \text{grad } T \right) +$$

$$+ C_{\text{gw}}^{py_c} \left(\frac{\partial y_c}{\partial t} + \mathbf{v}_* \cdot \text{grad } y_c \right) + \text{div} [\eta (\varrho_c - \varrho_w) \mathbf{J}_c^F] - C_{\text{gw}}^{pI_1} \left(\frac{\partial I_1}{\partial t} + \mathbf{v}_* \cdot \text{grad } I_1 \right) = S_{\text{gw}},$$

Heat transfer

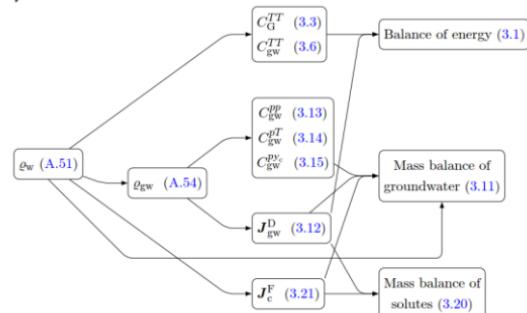
Diffusion $\mathbf{J}_{\text{gw}}^D = - (\mathbf{K}_{\text{gw}}^{pp} \cdot \text{grad } p - \mathbf{K}_{\text{gw}} \cdot \varrho_{\text{gw}} \mathbf{g} + \mathbf{K}_{\text{gw}}^{pT} \cdot \text{grad } T).$

Solute transport

Bedrock deformation

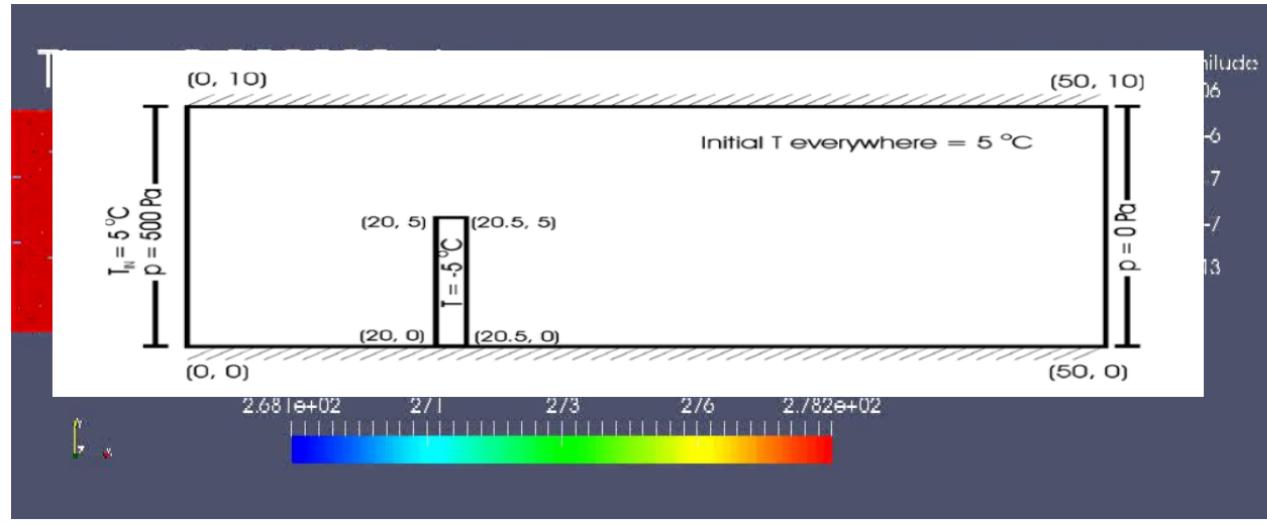
- High level of mutual dependencies (direct/indirect)

$$\begin{aligned} \varrho_w = \varrho_{w0} \exp \left[\kappa_{w0} (p - p_0) - \alpha_{w0} (T - T_0) \left(a_{\alpha_w,0} + \frac{1}{2} a_{\alpha_w,1} \frac{T - T_0}{T_0} + \frac{1}{3} a_{\alpha_w,2} \left(\frac{T - T_0}{T_0} \right)^2 + \right. \right. \\ \left. \left. + \frac{1}{4} a_{\alpha_w,3} \left(\frac{T - T_0}{T_0} \right)^3 + \frac{1}{5} a_{\alpha_w,4} \left(\frac{T - T_0}{T_0} \right)^4 + \frac{1}{6} a_{\alpha_w,5} \left(\frac{T - T_0}{T_0} \right)^5 \right) + \right. \\ \left. + \zeta_{w0} (x_w - x_{w0}) \left(b_{\zeta_w,0} + \frac{1}{2} b_{\zeta_w,1} (x_w - x_{w0}) \right) \right], \quad (\text{A.51}) \end{aligned}$$



Validation of single components

Coupled groundwater flow (after McKenzie et al., 2007)



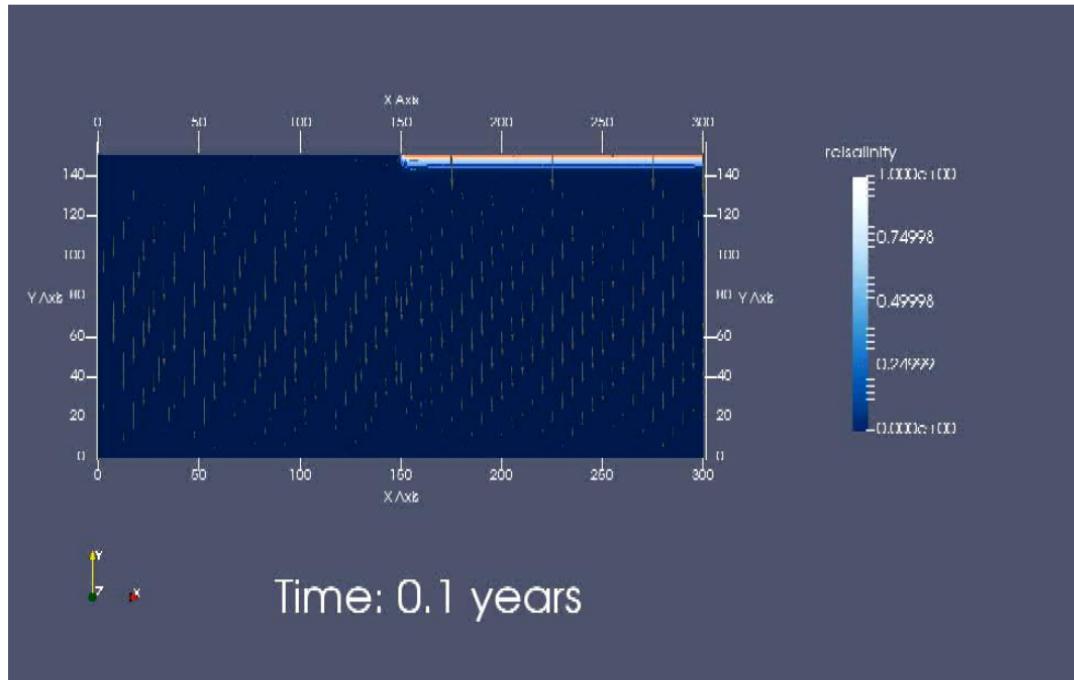
Groundwater
transport

Heat transfer

Simplified version of this experiment in source code: [elmerfem/elmerice/Tests/Permafrost_Frozenwall](https://github.com/ElmerFEM/elmerice/tree/main/Tests/Permafrost_Frozenwall)

Validation of single components

Elder Problem (Voss and Souza, 1987): salinity transport in porous medium



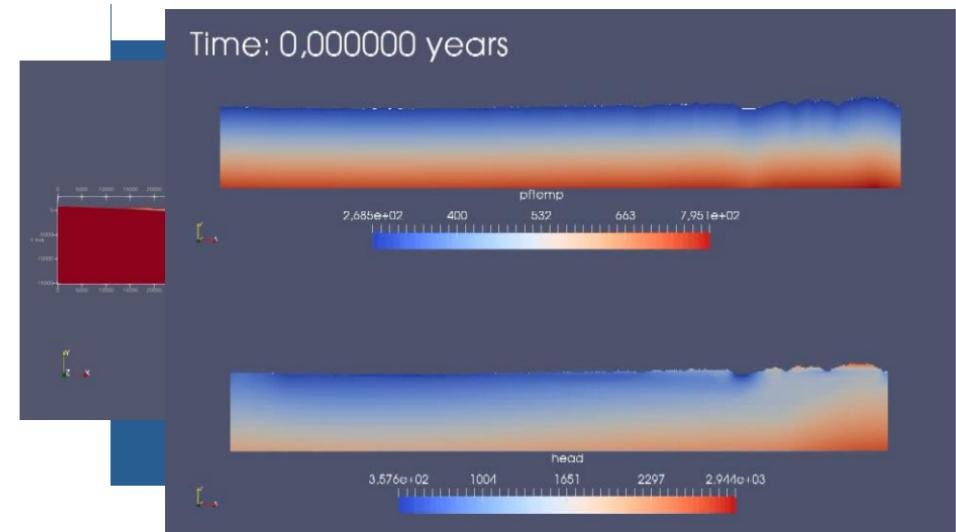
Groundwater
transport

Salinity
transport



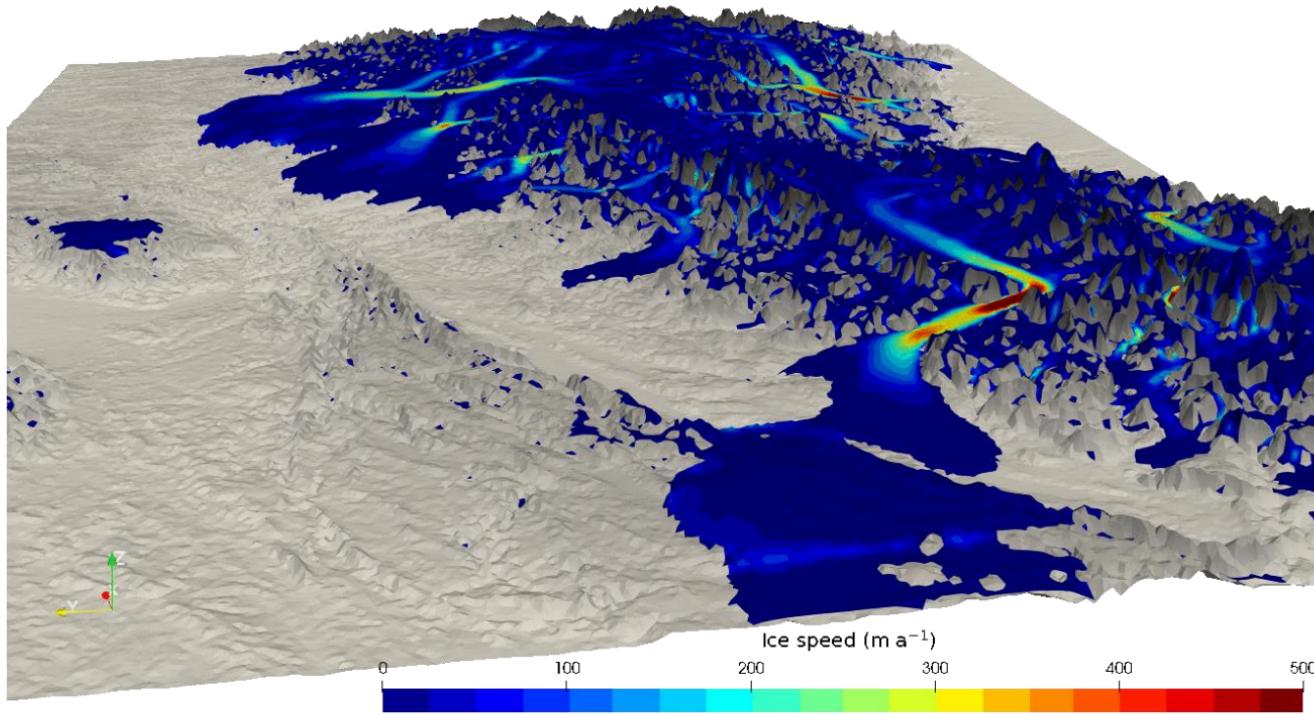
Nuclear waste repository safety assessment: Rhine Glacier

- Rhine Glacier simulation
- Flow-line along main branch
- Highly complex geology



nagra.

Nuclear waste repository safety assessment: Rhine Glacier

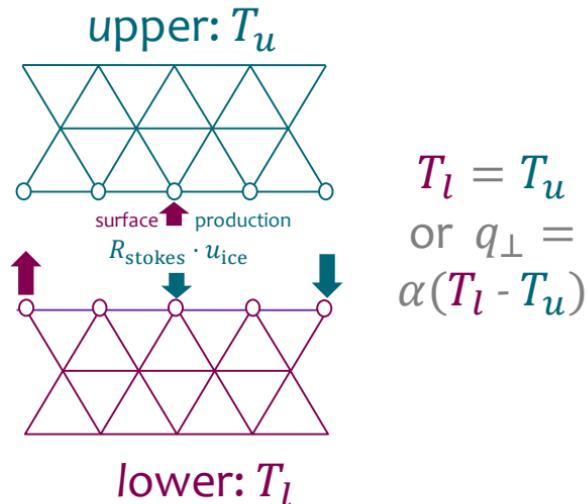


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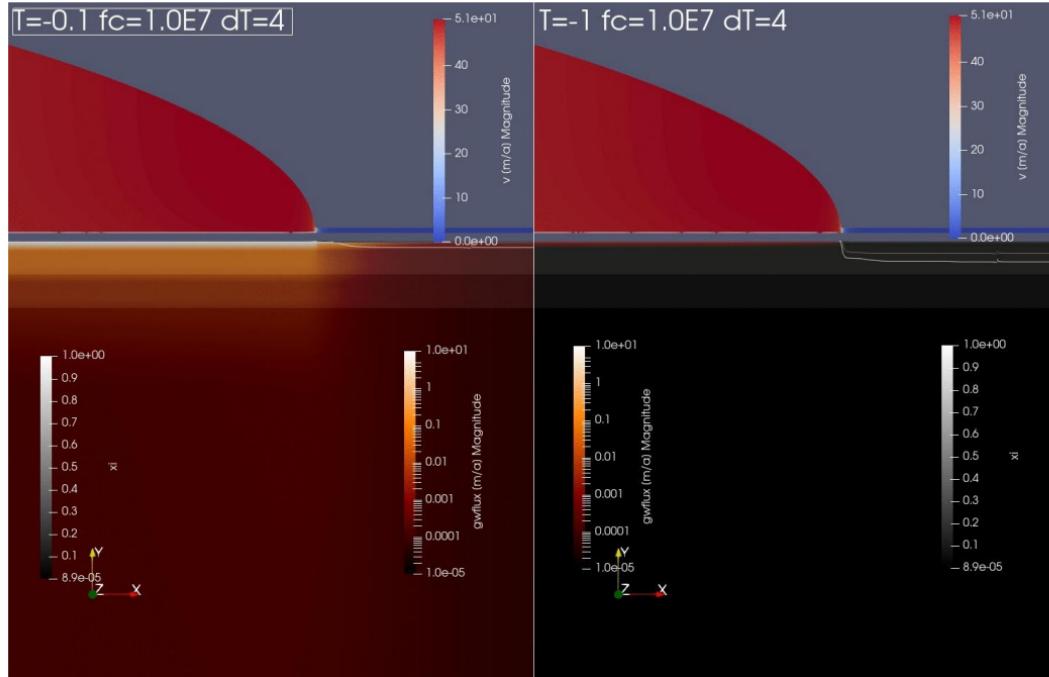
Coupling of ice-sheet to permafrost

- Coupling of solver "of same kind" (e.g. Stokes and lin. Elasticity; HTEQ in ice and permafrost)
- Either Dirichlet-Neumann or Robin-Neumann
- Elegantly using residual as load
- Can also include surface production term

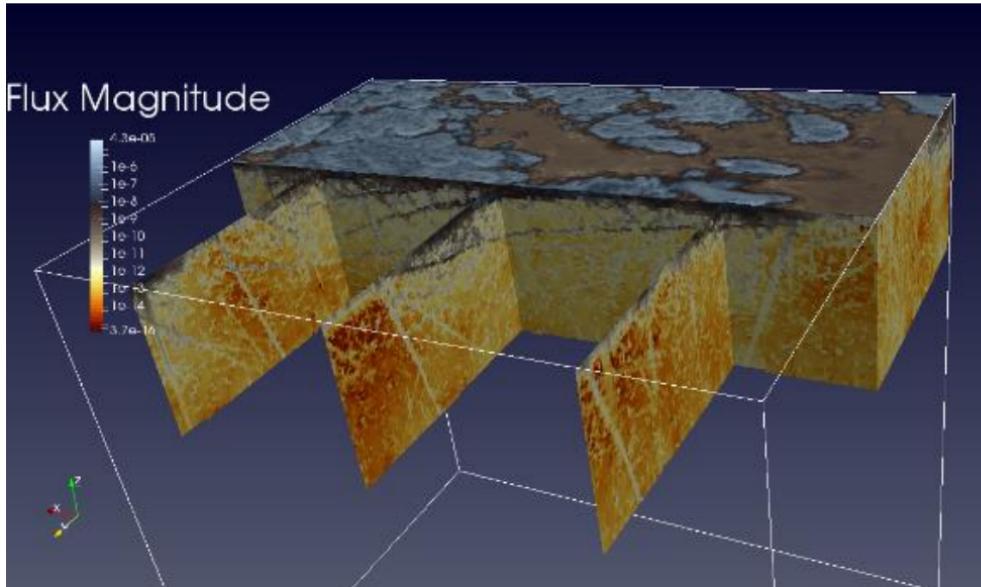
$$\text{Load}_u = R_l$$



Coupling of ice-sheet to permafrost



Nuclear waste repository safety assessment: Olkiluoto

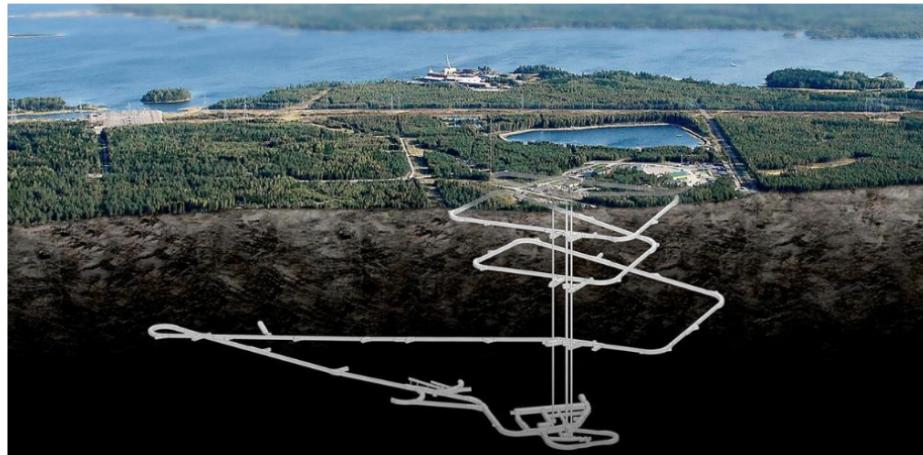


Posiva
Global leader
in final disposal

 **Elmer**

- Need to know future possible permafrost situation
- Development of high-fidelity groundwater/permafrost model
- $10 \times 10 \times 10 \text{ km}^3$ geological model around Olkiluoto – 30 m^3 resolution
- Outlook: include effects of approaching ice-sheet margin

Nuclear waste repository safety assessment: Olkiluoto



Source: Posiva.

- Need to know future possible permafrost situation
- Development of high-fidelity groundwater/permafrost model
- $10 \times 10 \times 10 \text{ km}^3$ geological model around Olkiluoto – 30 m^3 resolution
- Outlook: include effects of approaching ice-sheet margin

Nuclear waste repository safety assessment: Olkiluoto

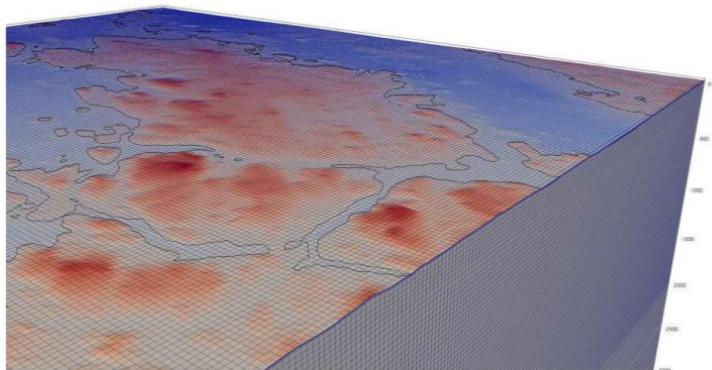


Figure 3: North-western corner of finite element mesh showing regular $30 \text{ m} \times 30 \text{ m} \times 30 \text{ m}$ discretization in the crystalline bedrock down to $z = -2010 \text{ m}$, coarser discretization below (bottom right corner of figure), discretization in the soil layers. Horizontal discretization is uniform at $30 \text{ m} \times 30 \text{ m}$.

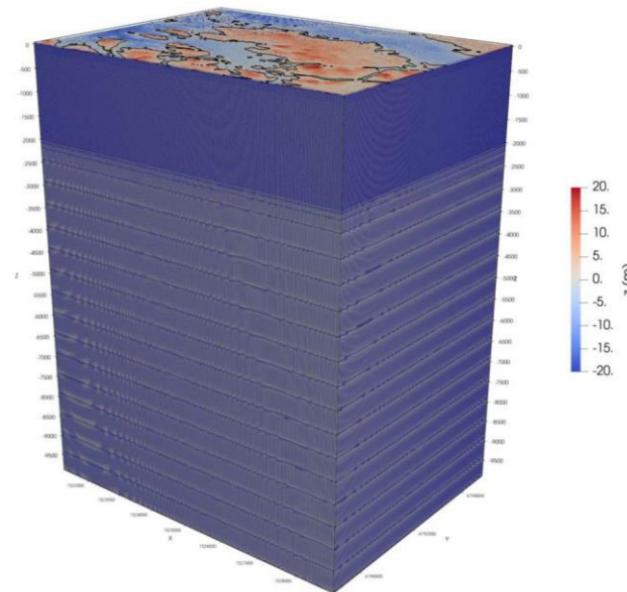


Figure 2: Three-dimensional model domain showing finite element mesh discretization.

Nuclear waste repository safety assessment: Olkiluoto

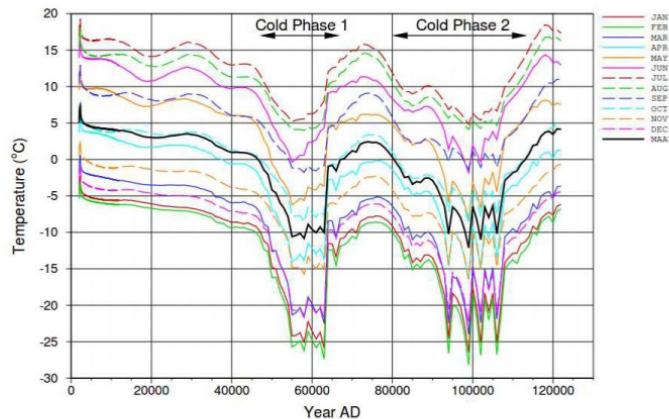


Figure 10: Time series of monthly averaged and mean annual (MAAT) 2-meter air temperature for RCP4.5 Minimum Sea Level Scenario. Also indicated are the two cold phases, Cold Phase 1 and Cold Phase 2, identified by a MAAT less than zero degree Celsius.

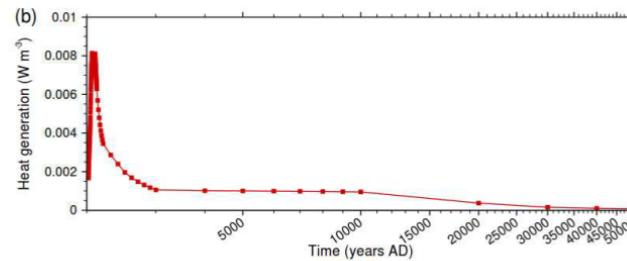


Figure 14: Heat generation at Olkiluoto for Power Case 1 plotted with two different time scales: (a) linear and (b) using a logarithmic time scale.

Nuclear waste repository safety assessment: Olkiluoto

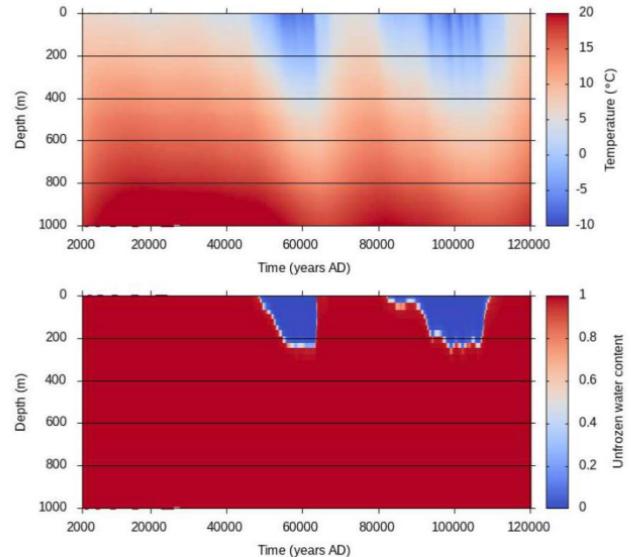


Figure 20: Evolution of (top) temperature and (bottom) unfrozen water content (χ) at site 1 (see Figure 19). EPSG:2391 coordinate 1,526,385 m, 6,792,340 m.

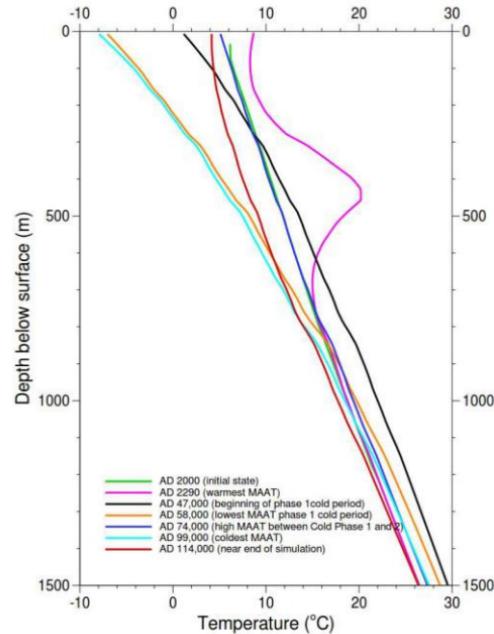


Figure 25: Vertical temperature profile for site 3 down to a depth of 1500 m at various times during the simulation.

Nuclear waste repository safety assessment: Olkiluoto

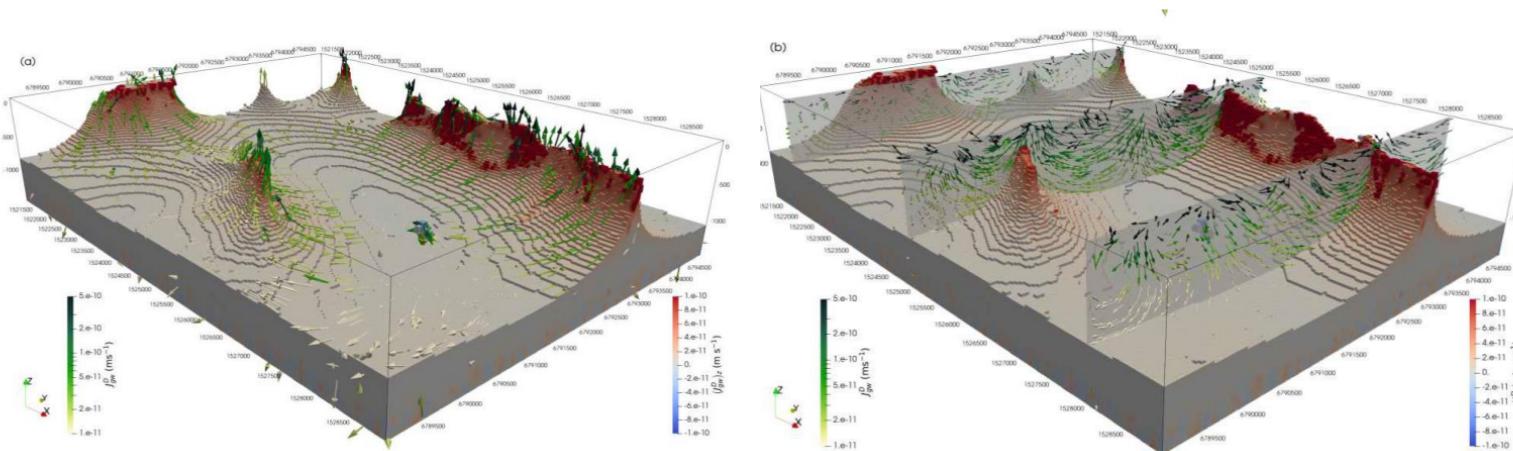


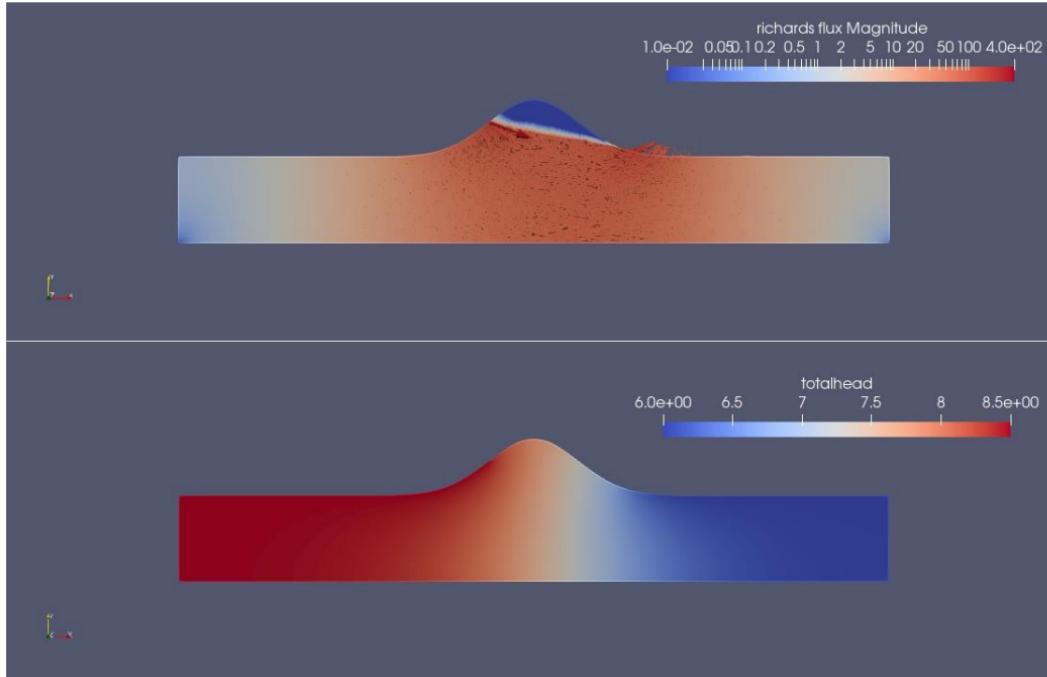
Figure 45: Three dimensional block view at AD 45,021 before permafrost development of isosurface $y_c = 2.0 \text{ g L}^{-1}$ showing groundwater flow vectors along (a) isosurface and (b) three vertical cross sections. Isosurface is colored according to vertical groundwater flux.

Nuclear waste repository safety assessment: Olkiluoto

The screenshot shows the Posiva website's 'Reports and publications' page. At the top, there is a navigation bar with links for Company, Final disposal, Responsibility, Career, News, Media, Partners, and Posiva Solutions. Below the navigation is a search bar with fields for Name / Content, Report type, Report language, ISBN / ISSN number, Report number, and a checkbox for 'Search from content'. There are also filters for Report author (Cohen), Report title, and Publishing date (Choose time range). A search button is located at the bottom of the search bar. Below the search bar, there is a heading 'Cohen' with an 'X' icon. Underneath, there is a table header with columns for Name, File count, File size, File format, Published, and Modified. The table lists one item: 'Working Report 2021-14 Modeling Permafrost Evolution at Olkiluoto for the Next 120 000 Years' by Cohen-Corticchiato, D., Zwinger, T., published on 05.07.2021. There are 'Download (79.7 MB)' and 'Open' buttons next to the report entry.

Name	File count	File size	File format	Published	Modified
Working Report 2021-14 Modeling Permafrost Evolution at Olkiluoto for the Next 120 000 Years				05.07.2021	

Richards equation for partly saturated aquifer (van Genuchten)



See Model 69 in Elmer
Models Manual

<https://www.nic.funet.fi/pub/sci/physics/elmer/doc/ElmerModelsManual.pdf>



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linkedin.com/company/csc--it-center-for-science



github.com/CSCfi

Additional material



Permafrost model

$$(T, p, y_c, \eta)$$

- Heat Transfer:

$$C_G^{TT} \left(\frac{\partial T}{\partial t} + \mathbf{v}_* \cdot \nabla T \right) + C_{gw}^{TT} \nabla T \cdot \mathbf{J}_{gw}^D + \operatorname{div} \mathbf{J}_G^H + \\ + C_G^{Tp} \left(\frac{\partial p}{\partial t} + \mathbf{v}_* \cdot \nabla p \right) + C_G^{Ty_c} \left(\frac{\partial T}{\partial t} + \mathbf{v}_* \cdot \nabla y_c \right) = S_G$$

$$C_{gw}^{pp} \left(\frac{\partial p}{\partial t} + \mathbf{v}_* \cdot \nabla p \right) + \operatorname{div} (\varrho_{gw} \mathbf{J}_{gw}^D) + C_{gw}^{pT} \left(\frac{\partial T}{\partial t} + \mathbf{v}_* \cdot \nabla T \right) + \\ + C_{gw}^{py_c} \left(\frac{\partial y_c}{\partial t} + \mathbf{v}_* \cdot \nabla y_c \right) + \operatorname{div} [\eta (\varrho_c - \varrho_w) \mathbf{J}_c^F] - C_{gw}^{pI_1} \left(\frac{\partial I_1}{\partial t} + \mathbf{v}_* \cdot \nabla I_1 \right) = S_{gw}$$

- Groundwater flow:

$$C_c^{y_c y_c} \left(\frac{\partial y_c}{\partial t} + \mathbf{v}_* \cdot \nabla y_c \right) + \operatorname{div} \left(\frac{y_c}{\chi} \varrho_c \mathbf{J}_{gw}^D \right) + \operatorname{div} (\eta \varrho_c \mathbf{J}_c^F) + \\ + C_c^{y_c T} \left(\frac{\partial T}{\partial t} + \mathbf{v}_* \cdot \nabla T \right) + C_c^{y_c p} \left(\frac{\partial p}{\partial t} + \mathbf{v}_* \cdot \nabla p \right) = S_c$$

- Solute transport:

$$-\operatorname{div} (\boldsymbol{\sigma}^{\text{ef}} - p \mathbf{I}) = \rho_G \mathbf{g},$$

$$\boldsymbol{\sigma}^{\text{ef}} = \mathbb{C}_G : \boldsymbol{\epsilon} - \beta_G (T - T_0)$$

Permafrost model

- Heat flux:

$$\mathbf{J}_G^H = -\mathbf{K}_G^{TT} \cdot \text{grad } T$$

$$\mathbf{J}_{gw}^D = -(\mathbf{K}_{gw}^{pp} \cdot \text{grad } p - \mathbf{K}_{gw} \cdot \varrho_{gw} \mathbf{g} + \mathbf{K}_{gw}^{pT} \cdot \text{grad } T).$$

- Groundwater flux:

$$\mathbf{J}_c^F = -(\mathbf{K}_c^{y_c y_c} \cdot \text{grad } y_c - \mathbf{K}_c \cdot \mathbf{f}_c y_c)$$

- Solute flux:

- Elasticity tensor:

$$\mathbf{K}_G^{uu} = \frac{E_G}{(1+\nu_G)(1-2\nu_G)} \begin{bmatrix} 1-\nu_G & \nu_G & \nu_G & 0 & 0 & 0 \\ \nu_G & 1-\nu_G & \nu_G & 0 & 0 & 0 \\ \nu_G & \nu_G & 1-\nu_G & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2\nu_G}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\nu_G}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\nu_G}{2} \end{bmatrix},$$

$$E_G = (1-\eta) \frac{E_{s,0}}{1-\eta_0} + \eta(1-\chi) E_i,$$

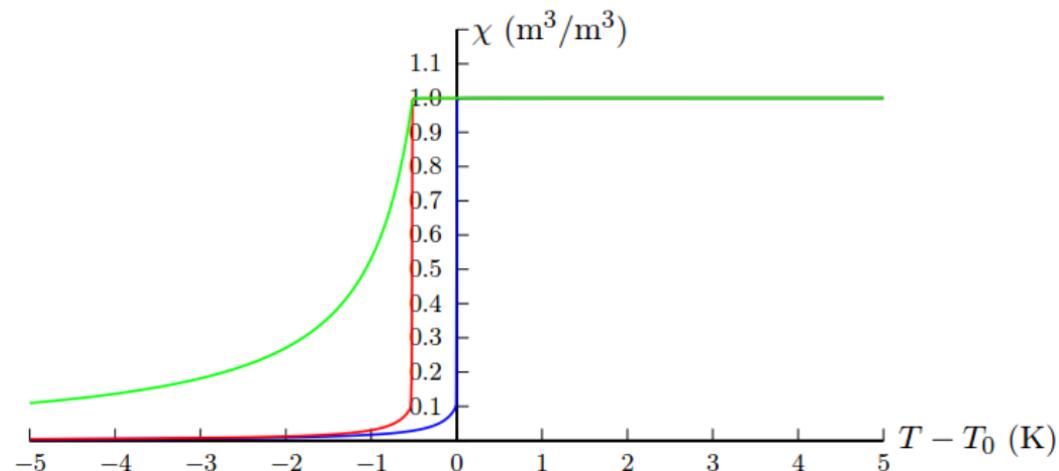
$$\nu_G = (1-\eta)\nu_{s,0} + \eta(1-\chi)\nu_i.$$

Permafrost model

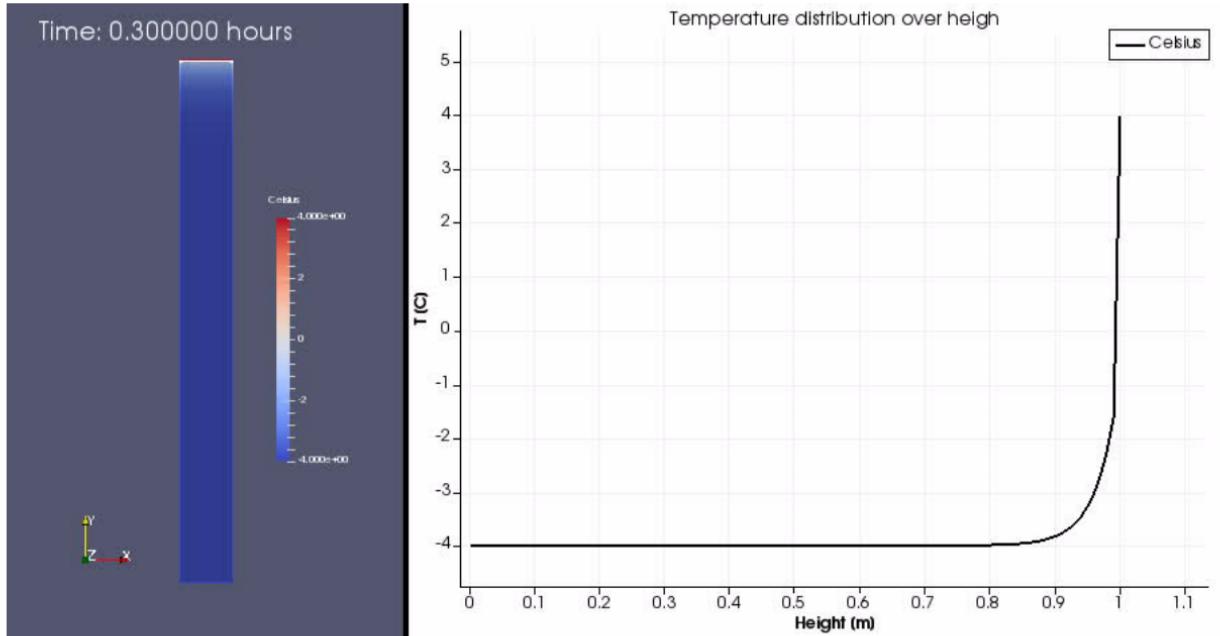
- Phase change: Frozen water content $[0,1]$

$$\chi = \chi(T, p, y_c, \eta)$$

$$y_c = \chi \cdot x_c,$$



Validation of single components



Heat transfer

Stefan Problem: freezing/thawing front in water column